

Non-connective K -theory of relative exact categories

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Abstract

The main objective of this paper is to propose a definition of non-connective K -theory for a wide class of relative exact categories which, in general, do not satisfy the factorization axiom and confirm that it agrees with the non-connective K -theory for exact categories and complicial exact categories with weak equivalences. The main application is to study the topological filtrations of non-connective K -theory of a noetherian commutative ring with unit in terms of Koszul cubes.

Introduction

As in [Sch04], [Sch06] and [Sch11], Schlichting developed the non-connective K -theory for the wide class of Waldhausen exact categories, and the non-connective K -theory for differential graded categories and for stable infinity categories are characterized by D.-C. Cisikins and G. Tabuada, A.J. Blumberg and D. Gepner and G. Tabuada in [CT11] and [BGT13] respectively. This generalizes the definition of Bass [Bas68], Karoubi [Kar70], Pedersen-Weibel [Ped84], [PW89], Thomason [TT90], Carter [Car80] and Yao [Yao92]. My motivational theme is to study the topological filtrations of non-connective K -theory of a noetherian commutative ring with unit in terms of Koszul cubes in [Moc13]. As precisely mentioned in Remark 8.20, the biWaldhausen category of Koszul cubes does not satisfy the factorization axiom in [Sch06] and the first purpose of this paper is to establish a general theory about non-connective K -theory for a certain wide class of Waldhausen exact categories which, in general, need not satisfy the factorization axiom.

Let $\mathbf{E} = (\mathcal{E}, w)$ be a **relative exact category**, that is, a pair of an exact category \mathcal{E} with a specific zero object 0 and a class of morphisms w in \mathcal{E} which is closed under finite compositions. (See Definition 2.2). We let \mathcal{E}^w denote the full subcategory of \mathcal{E} consisting of those objects x such that the canonical morphism from the zero object $0 \rightarrow x$ is in w . We say that \mathbf{E} is **strict** if \mathcal{E}^w is an exact category such that the inclusion functor $\mathcal{E}^w \hookrightarrow \mathcal{E}$ is exact and reflects exactness. (See Ibid). For example, \mathbf{E} is strict if either w satisfies the extensional axiom or \mathbf{E} is a Waldhausen exact category. (See Proposition 2.4). We denote the bounded derived category of an exact category \mathcal{F} by $\mathcal{D}_b(\mathcal{F})$. We shall define the bounded derived category of a strict relative exact category $\mathbf{E} = (\mathcal{E}, w)$ by the formula $\mathcal{D}_b(\mathbf{E}) := \text{Coker}(\mathcal{D}_b(\mathcal{E}^w) \rightarrow \mathcal{D}_b(\mathcal{E}))$. (See Definition 2.5). Let $j_{\mathcal{E}} : \mathcal{E} \rightarrow \mathbf{Ch}_b(\mathcal{E})$ denote the exact functor which sends an object x to a complex $j_{\mathcal{E}}(x)$ such that $j_{\mathcal{E}}(x)_k$ is x if $k = 0$ and is 0 if $k \neq 0$ and we write $P_{\mathbf{E}} : \mathbf{Ch}_b(\mathcal{E}) \rightarrow \mathcal{D}_b(\mathbf{E})$ for the canonical projection functor. We say that a morphism $f : x \rightarrow y$ in $\mathbf{Ch}_b(\mathcal{E})$ is a **quasi-weak equivalence** if $P_{\mathcal{E}}(f)$ is an isomorphism in $\mathcal{D}_b(\mathbf{E})$. We write qw for the class of quasi-weak equivalences in $\mathbf{Ch}_b(\mathcal{E})$ and we put $\mathbf{Ch}_b(\mathbf{E}) := (\mathbf{Ch}_b(\mathcal{E}), qw)$. For example, if w is the class of all isomorphisms in \mathcal{E} , then qw is just the class of all quasi-isomorphisms in $\mathbf{Ch}_b(\mathcal{E})$. $\mathbf{Ch}_b(\mathbf{E})$ is a complicial exact category with weak equivalences in the sense of [Sch11]. (See Proposition 3.20). We say that a strict relative exact category $\mathbf{E} = (\mathcal{E}, w)$ is a **consistent** relative

exact category if $j_{\mathcal{E}}(w) \subset qw$. (See Lemma-Definition 2.9). We will build the universal property of $\mathbf{Ch}_b(\mathbf{E})$ for any consistent relative exact category \mathbf{E} in Corollary 4.11 which vouches for the pedigree of the operation $\mathbf{Ch}_b(-)$. The first main theorem below also warrants $\mathbf{Ch}_b(\mathbf{E})$ to be a natural object.

Theorem 0.1 (Derived Gillet-Waldhausen theorem). (A part of Corollary 4.15). *For any consistent relative exact category $\mathbf{E} = (\mathcal{E}, w)$, the canonical functor $j_{\mathcal{E}} : \mathbf{E} \rightarrow \mathbf{Ch}_b(\mathbf{E})$ induces an equivalence of triangulated categories $\mathcal{D}_b(\mathbf{E}) \xrightarrow{\sim} \mathcal{D}_b(\mathbf{Ch}_b(\mathbf{E}))$.*

The theorem roughly says that the process of taking $\mathbf{Ch}_b(-)$ does not change the matters up to derived equivalences (See also Corollary 4.16) and encourages us to define the non-connective K -theory of a consistent relative exact category $\mathbf{E} = (\mathcal{E}, w)$ by the formula $\mathbb{K}(\mathbf{E}) = \mathbb{K}^S(\mathbf{Ch}_b(\mathbf{E}))$. (See Definition 3.22). Here \mathbb{K}^S means the Schlichting non-connective K -theory in [Sch06] or [Sch11]. Theorem 0.1 and Schlichting theory in [Sch11] imply that the non-connective K -theory of consistent relative exact categories is a *localizing theory* as in Corollary 0.2 below. We say that a sequence of triangulated categories $\mathcal{T} \xrightarrow{i} \mathcal{T}' \xrightarrow{p} \mathcal{T}''$ is **weakly exact** if pi is isomorphic to the zero functor, i is fully faithful and the induced functor $\mathcal{T}' / \mathcal{T} \rightarrow \mathcal{T}''$ is **cofinal**. The last condition means that it is fully faithful, and every object in \mathcal{T}'' is a direct summand of an object of $\mathcal{T}' / \mathcal{T}$.

Corollary 0.2. *For a sequence of consistent relative exact categories $\mathbf{E} \rightarrow \mathbf{F} \rightarrow \mathbf{G}$, if the induced sequence of triangulated categories $\mathcal{D}_b(\mathbf{E}) \rightarrow \mathcal{D}_b(\mathbf{F}) \rightarrow \mathcal{D}_b(\mathbf{G})$ is weakly exact, then the sequence induces a fibration sequence of spectra*

$$\mathbb{K}(\mathbf{E}) \rightarrow \mathbb{K}(\mathbf{F}) \rightarrow \mathbb{K}(\mathbf{G}).$$

In particular if the induced morphism $\mathcal{D}_b(\mathbf{E}) \rightarrow \mathcal{D}_b(\mathbf{F})$ is an equivalence of triangulated categories up to factor, then the induced morphism $\mathbb{K}(\mathbf{E}) \rightarrow \mathbb{K}(\mathbf{F})$ is a homotopy equivalence of spectra.

A proof of Corollary 0.2 will be given at 4.18. Next, if $\mathbf{E} = (\mathcal{E}, w)$ is a Waldhausen exact category, we can also define $K^W(\mathbf{E}) = K^W(\mathcal{E}; w)$ the Waldhausen K -theory of \mathbf{E} . There is a question of what is a sufficient conditions that $j_{\mathcal{E}} : \mathbf{E} \rightarrow \mathbf{Ch}_b(\mathbf{E})$ induces an isomorphism $K_n^W(\mathbf{E}) \rightarrow K_n^W(\mathbf{Ch}_b(\mathbf{E})) = \mathbb{K}_n^S(\mathbf{Ch}_b(\mathbf{E})) = \mathbb{K}_n(\mathbf{E})$ for any positive integer n . We will assay this problem by axiomatic approach in section 7 and carve out the agreement result Theorem 0.3 below. To state the theorem, we prepare or recall the notations. A strict relative exact category $\mathbf{E} = (\mathcal{E}, w)$ is **solid** if for any morphism $f : x \rightarrow y$ in \mathcal{E} , there is a zig-zag sequence of quasi-isomorphisms connecting the mapping cone $\text{Cone } f = [x \xrightarrow{f} y]$ with a bounded complex in \mathcal{E}^w . (See Lemma-Definition 4.12). We say that a strict relative exact category \mathbf{E} is **very strict** if the inclusion functor $\mathcal{E}^w \hookrightarrow \mathcal{E}$ induces a fully faithful functor on the bounded derived categories $\mathcal{D}_b(\mathcal{E}^w) \hookrightarrow \mathcal{D}_b(\mathcal{E})$. (See Definition 2.2). For example, if either w is the class of all isomorphisms in \mathcal{E} or \mathbf{E} is a complicial exact category with weak equivalences, then \mathbf{E} is very strict and solid. (See Proposition 3.9 and Corollary 4.13). Let $\mathbf{C} = (\mathcal{C}, v)$ be a category with cofibrations and weak equivalences. We denote the Waldhausen K -theory of \mathbf{C} by $K^W(\mathbf{C}) = K^W(\mathcal{C}; v)$. If v is the class of all isomorphisms in \mathcal{C} , we shortly write $K^W(\mathcal{C})$ for $K^W(\mathcal{C}; v)$. We say that $\mathbf{C} = (\mathcal{C}, v)$ satisfies the **K^W -fibration axiom** if $\mathcal{C}^v \hookrightarrow \mathcal{C}$ is a subcategory with cofibrations and if the inclusion functor $\mathcal{C}^v \hookrightarrow \mathcal{C}$ and the identity functor of \mathcal{C} induce a fibration sequence of spectra $K^W(\mathcal{C}^v) \rightarrow K^W(\mathcal{C}) \rightarrow K^W(\mathcal{C}; v)$. (See Lemma-Definition 7.4). It is well-known that if v satisfies the extensional, saturated and factorization axioms, then \mathbf{C} satisfies the K^W -fibration axiom. (See [Sch06, Theorem 11]). We have the following agreement results.

Theorem 0.3 (Agreement). *Let $\mathbf{E} = (\mathcal{E}, w)$ is a consistent relative exact category. Then*
 (1) **(Agreement with Grothendieck groups).** $\mathbb{K}_0(\mathbf{E})$ *is isomorphic to the Grothendieck group*

of the idempotent completion of the triangulated category $\mathcal{D}_b(\mathbf{E})$.

(2) **(Agreement with Schlichting K -theory).** If either w is the class of all isomorphisms in \mathcal{E} or \mathbf{E} is a complicial exact category with weak equivalences, then the exact functor $j_{\mathcal{E}} : \mathbf{E} \rightarrow \mathbf{Ch}_b(\mathbf{E})$ induces a homotopy equivalences of spectra $\mathbb{K}^S(\mathbf{E}) \xrightarrow{\sim} \mathbb{K}(\mathbf{E})$.

(3) **(Agreement with Waldhausen K -theory).** If \mathbf{E} is a very strict solid Waldhausen exact category which satisfies the K^W -fibration axiom, then for any positive integer n , the n -th homotopy group of $\mathbb{K}(\mathbf{E})$ is isomorphic to the n -th Waldhausen K -theory $K_n^W(\mathbf{E})$ of \mathbf{E} .

A proof of Theorem 0.3 will be given in 7.7. The second purpose of this paper is to generalize the results in [Moc13] from the connective K -theory to the non-connective K -theory.

Let us fix a commutative noetherian ring with unit A , a finite set S and a family of elements $\mathbf{f}_S = \{f_s\}_{s \in S}$ in A which is an A -regular sequence in any order. Let us denote the power set of S by $\mathcal{P}(S)$, the set of all subsets in S with usual inclusion order. A **Koszul cube** x associated with a sequence $\mathbf{f}_S = \{f_s\}_{s \in S}$ is a contravariant functor from $\mathcal{P}(S)$ to the category of finitely generated projective A -modules such that for any subset T of S and any element k in T , $d_T^k := x(T \setminus \{k\} \hookrightarrow T)$ is an injection and $f_k^{m_k} \text{Coker } d_T^k = 0$ for some m_k . We denote the category of Koszul cubes associated with \mathbf{f}_S by $\mathbf{Kos}_A^{\mathbf{f}_S}$ where morphisms between Koszul cubes are natural transformations. (See Definition 9.1). We let $\mathbf{Perf}_{\text{Spec } A}^{V(\mathbf{f}_S)}$ denote the category of perfect complexes on $\text{Spec } A$ whose homological support is in $V(\mathbf{f}_S)$. We denote the class of all quasi-isomorphisms in $\mathbf{Perf}_{\text{Spec } A}^{V(\mathbf{f}_S)}$ by qis . There exists the exact functor $\text{Tot} : \mathbf{Kos}_A^{\mathbf{f}_S} \rightarrow \mathbf{Perf}_{\text{Spec } A}^{V(\mathbf{f}_S)}$. We define the **class of total quasi-isomorphisms** by pull-back of qis in $\mathbf{Perf}_{\text{Spec } A}^{V(\mathbf{f}_S)}$ by Tot and denote it by tq . (See Definition 8.9 and 9.3). Theorem 0.4 below together with Theorem 0.3 (3) convince us that the non-connective K -theory of the relative exact category $(\mathbf{Kos}_A^{\mathbf{f}_S}, \text{tq})$ accords with the Waldhausen K -theory of it.

Theorem 0.4 (A part of Corollary 9.6). $(\mathbf{Kos}_A^{\mathbf{f}_S}, \text{tq})$ is a very strict solid Waldhausen exact category which satisfies the K^W -fibration axiom.

The key ingredient to figure out the structure of $(\mathbf{Kos}_A^{\mathbf{f}_S}, \text{tq})$ is the existence of the flag structure (See Definition 6.11) on $(\mathbf{Kos}_A^{\mathbf{f}_S}, \text{tq})$ and this fact is verified by polishing up results in [Moc13]. The final main theorem below is the comparison theorem referred in the Abstract.

Theorem 0.5 (Weak geometric presentation theorem). The exact functor $\text{Tot} : (\mathbf{Kos}_A^{\mathbf{f}_S}, \text{tq}) \rightarrow (\mathbf{Perf}_{\text{Spec } A}^{V(\mathbf{f}_S)}, \text{qis})$ induces an equivalence of the bounded derived categories $\mathcal{D}_b(\mathbf{Kos}_A^{\mathbf{f}_S}, \text{tq}) \rightarrow \mathcal{D}_b(\mathbf{Perf}_{\text{Spec } A}^{V(\mathbf{f}_S)}, \text{qis})$. In particular it also induces a homotopy equivalence of spectra $\mathbb{K}(\mathbf{Kos}_A^{\mathbf{f}_S}, \text{tq}) \rightarrow \mathbb{K}(\mathbf{Perf}_{\text{Spec } A}^{V(\mathbf{f}_S)}, \text{qis})$.

A proof of Theorem 0.5 will be given at 9.7. In my subsequent paper, we will sophisticate Theorem 0.5 as a comparison of two different kinds of weights immanent in a scheme from the viewpoint of non-commutative motive theory. It is the reason where the term “geometric presentation” comes from. Now we give a guide for the structure of this paper. To demonstrate Theorem 0.1, full techniques evolved from section 1 to section 4 are required. Under the influence of support varieties theory, like as [Bal07] and [BKS07], in this paper we accentuate lattice structures of subcategories in appropriate classes of categories. We will probe the lattice structures of (thick) triangulated subcategories of triangulated categories in section 1 and of null classes of bicomplial categories in section 3. These observations leads us to characterization of quasi-weak equivalences in terms of total weak equivalences in section 4. Section 2 and section 6 are devoted to glossaries of relative exact categories and quasi-split

exact sequences respectively. The contents in section 6 is related with [FL12]. As pointed out in [Sch11, A.2.8], quasi-split exact sequences of triangulated categories are a variations of Bousfield-Neeman localization theory unfolded in [Nee01, §9]. In section 5, we will sum up resolution theorems for relative exact categories from [Sch11] and [Moc13]. As a sequel to [Moc13], we will inquire into derived categories and derived flags of multi semi-direct products of exact categories in section 8 and of Koszul cubes in section 9. As a matter of my principle, the establishment of motive theory for a wide class of relative exact categories including the relative exact categories of Koszul cubes is a gateway to decipher motivic interpretations of Gersten's conjecture in [Ger73] and Weibel's K -dimensional conjecture in [Wei80]. (They are theorems in many situations). Section 7 covers general additive and localizing theories over relative exact categories like as companion theories over DG-categories or infinity-categories in [Tab08] or [BGT13]. (See also Corollary 4.16). These results allow us to apply knowledge of Koszul cubes in section 9 to not only the non-connective K -theory but also any other localizing theories. We conclude with the remark that there is a strong resemblance between the flags of the relative exact categories of perfect complexes on projective spaces in Example 6.12 and of Koszul cubes in Corollary 9.6. In a sequel work, I hope to tie up Grayson-Walker weight complexes in [Gra95] and [Wal00] with Koszul cubes by utilizing the blow up formula in [Tho93] or [CHSW08].

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Conventions.

(1) General assumptions

Throughout the paper, we use the letters A , \mathcal{A} and S to denote a commutative ring with 1, an essentially small abelian category and a set respectively.

(2) Set theory

- (i) We denote the cardinality of a set S by $\#S$.
- (ii) For any pair of sets S and T , we put $S \setminus T := \{x \in S; x \notin T\}$. If S and T are disjoint, then we write $S \sqcup T$ for the union $S \cup T$ of S and T .
- (iii) For any set S , we write $\mathcal{P}(S)$ for its power set. Namely $\mathcal{P}(S)$ is the set of all subsets of S . We consider $\mathcal{P}(S)$ to be a partially ordered set under inclusion.

(3) Partially ordered sets

- (i) For any elements a and b in a partially ordered set P , we write $[a, b]$ for the set of all elements u in P such that $a \leq u \leq b$. We consider $[a, b]$ as a partially ordered subset of P if $a \leq b$ and $[a, b] = \emptyset$ if otherwise. We often use this notation to any integers a and b .
- (ii) For any non-negative integer n and any positive integer m , we denote $[0, n]$ and $[1, m]$ by $[n]$ and $(m]$ respectively.
- (iii) The **trivial ordering** \leq on a set S is defined by $x \leq y$ if and only if $x = y$.
- (iv) An element x in a partially ordered set P is **maximal** (resp. **minimal**) if for any element a in P , the inequality $x \leq a$ (resp. $a \leq x$) implies the equality $x = a$. An element x in a partially ordered set P is **maximum** (resp. **minimum**) if the inequality $a \leq x$ (resp. $x \leq a$) holds for any element a in P .
- (v) We say that a partially ordered set L is a **lattice** (resp. **\vee -complete**, resp. **\wedge -complete**, resp. a **complete lattice**) if for any elements x and y (resp. non-empty subset S) in L , there exists both $\sup\{x, y\}$ and $\inf\{x, y\}$ (resp. $\sup S$, resp. $\inf S$, resp. both $\sup S$ and $\inf S$) in L . We write $x \vee y$ and $x \wedge y$ for $\sup\{x, y\}$ and $\inf\{x, y\}$ respectively and call them the **join** and **meet** of x and y respectively. We also denote $\sup S$ (resp. $\inf S$) by $\bigvee_{u \in S} u$ or $\vee S$ (resp. $\bigwedge_{u \in S} u$ or $\wedge S$). In particular, if L is \vee -complete (resp. \wedge -complete) the maximum (resp. minimum) element

$1_L = \sup L$ (resp. $0_L = \inf L$) exists. We write **CLat** for the full subcategory of complete lattices in the category of partially ordered sets.

(4) Commutative algebra

- (i) We write A^\times for the group of units in A .
- (ii) If $f_S = \{f_s\}_{s \in S}$ is a subset of A , we write the same symbol f_S for the ideal they generate. By convention, we set $f_\emptyset = (0)$.
- (iii) For any ideal I of A and a letter p which is a natural number or ∞ , we let $\mathcal{M}_A^I(p)$ denote the category of finitely generated A -modules M of projective dimension $\leq p$ and whose support are in $V(I)$. We also put $\mathcal{P}_A = \mathcal{M}_A^{(0)}(0)$ and $\mathcal{M}_A = \mathcal{M}_A^{(0)}(\infty)$.

(5) Category theory

- (i) For any category \mathcal{X} , we denote the class of objects in \mathcal{X} by $\text{Ob } \mathcal{X}$ and for any objects x and y in \mathcal{X} , we write $\text{Hom}_{\mathcal{X}}(x, y)$ (or shortly $\text{Hom}(x, y)$) for the class of all morphisms from x to y . We say a category \mathcal{X} is **locally small** (resp. **small**) if for any objects x and y , $\text{Hom}_{\mathcal{X}}(x, y)$ forms a set (resp. if \mathcal{X} is locally small and $\text{Ob } \mathcal{X}$ forms a set).
- (ii) For any categories \mathcal{X} and \mathcal{Y} , we denote the (large) category of functors from \mathcal{X} to \mathcal{Y} by $\mathcal{Y}^{\mathcal{X}}$. Here the morphisms between functors from \mathcal{X} to \mathcal{Y} are just natural transformations. In particular, we write $\text{Ar } \mathcal{X}$ for $\mathcal{X}^{[1]}$ the category of morphisms in \mathcal{X} . There are canonical two functors $\text{dom}_{\mathcal{X}}$ and $\text{ran}_{\mathcal{X}}$ from $\text{Ar } \mathcal{X}$ to \mathcal{X} which send a morphism $f : x \rightarrow y$ in \mathcal{X} to x and y respectively. There is also a canonical natural transformation $\epsilon_{\mathcal{X}} : \text{dom} \rightarrow \text{ran}$ defined by $\epsilon(f) := f$ for any morphism $f : x \rightarrow y$ in \mathcal{X} .
- (iii) For any partially ordered set P , we regard it as a category P in the natural way. Namely, P is a category whose class of objects is P and for any elements x and y in P , $\text{Hom}_P(x, y)$ is the singleton $\{(x, y)\}$ if $x \leq y$ and is the emptyset \emptyset if otherwise. In particular, we regard any set S as a category by the trivial ordering on S .
- (iv) For any category \mathcal{X} , we denote the **dual category** of \mathcal{X} by \mathcal{X}^{op} . Namely $\text{Ob } \mathcal{X}^{\text{op}} = \text{Ob } \mathcal{X}$ and for any objects x and y in \mathcal{X} , $\text{Hom}_{\mathcal{X}^{\text{op}}}(x, y) := \text{Hom}_{\mathcal{X}}(y, x)$. Compositions of morphisms in \mathcal{X}^{op} is just same as in \mathcal{X} . In particular, for any partially ordered set P , P^{op} is said to be the **opposite partially ordered set** of P .
- (v) For any category \mathcal{X} , we write $i_{\mathcal{X}}$ or shortly i for the class of all isomorphisms in \mathcal{X} .
- (vi) Let \mathcal{C} be a category and \mathcal{S} a subclass of $\text{Ob } \mathcal{C}$. We write the same letter \mathcal{S} for the full subcategory \mathcal{X} in \mathcal{C} such that $\text{Ob } \mathcal{X} = \mathcal{S}$ and call it the **full subcategory (in \mathcal{C}) spanned by \mathcal{S}** .
- (vii) Let $\mathcal{C} \rightarrow \mathcal{C}'$ be a functor and \mathcal{S} a full subcategory of \mathcal{C}' . We write $f^{-1}\mathcal{S}$ for the full subcategory of \mathcal{C} spanned by $f^{-1}(\text{Ob } \mathcal{S})$ and call it the **pull-back of \mathcal{S} (by f)**.
- (viii) Let \mathcal{C} be a category and \mathcal{D} a full subcategory of \mathcal{C} and w a class of morphisms in \mathcal{C} . We write $\mathcal{D}_{w, \mathcal{C}}$ or simply \mathcal{D}_w for the full subcategory of \mathcal{C} consisting of those objects x such that there is a zig-zag sequence of morphisms in w connecting it to an object in \mathcal{D} . In particular, we write $\mathcal{D}_{\text{isom}, \mathcal{C}}$ or simply $\mathcal{D}_{\text{isom}}$ for $\mathcal{D}_{i_{\mathcal{C}}, \mathcal{C}}$ and call it the **isomorphisms closure of \mathcal{D} (in \mathcal{C})**. We say that a full subcategory \mathcal{D}' of \mathcal{C} is **tight** if $\mathcal{D}'_{\text{isom}, \mathcal{C}} = \mathcal{D}'$.
- (xi) We say that a morphism $p : y \rightarrow x$ in a category \mathcal{X} is a **retraction** or x is a **retraction** of y if there exists a morphism $i : x \rightarrow y$ such that $pi = \text{id}_x$. We say that a full subcategory \mathcal{D} of \mathcal{X} is **closed under retractions** if for any objects x and y in \mathcal{X} , if y is in \mathcal{D} and x is a retraction of y , then x is also in \mathcal{D} . We say that a class of morphisms w of \mathcal{X} satisfies the **retraction axiom** if w is closed under retractions in the morphism category of \mathcal{X} .
- (xii) A class of morphisms w in a category \mathcal{C} is a **multiplicative system** (resp. satisfies the **saturated axiom**) if w is closed under finite compositions and closed under isomorphisms (resp. for a pair of composable morphisms $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ in \mathcal{C} , if two of gf , g and f are in w , then the other one is also in w).
- (xiii) We mean that a **2-category** is a category of enriched by the category of (small) categories. For any objects x and y in a 2-category \mathcal{X} , we write $\mathcal{HOM}_{\mathcal{X}}(x, y)$ for the Hom category from x to y .

(xiv) A functor $f : \mathcal{X} \rightarrow \mathcal{Y}$ is **conservative** if it reflects isomorphisms, namely for any morphism $a : x \rightarrow y$ in \mathcal{X} , if fa is an isomorphism in \mathcal{Y} , then a is an isomorphism in \mathcal{X} .

(6) Relative categories

(i) We use the notations of relative categories theory from [BK12]. A **relative category** $\mathbf{C} = (\mathcal{C}, w)$ is a pair of a category \mathcal{C} and a class of morphisms w in \mathcal{C} such that w is closed under finite compositions. Namely if $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ are composable morphisms in w , then gf is also in w and id_x is in w for any object x in \mathcal{C} . A **relative functor** between relative categories $f : \mathbf{C} = (\mathcal{C}, w) \rightarrow \mathbf{C}' = (\mathcal{C}', w')$ is a functor $f : \mathcal{C} \rightarrow \mathcal{C}'$ such that $f(w) \subset w'$. A **relative natural equivalence** $\theta : f \rightarrow f'$ between relative functors $f, f' : \mathbf{C} = (\mathcal{C}, w) \rightarrow \mathbf{C}' = (\mathcal{C}', w')$ is a natural transformation $\theta : f \rightarrow f'$ such that $\theta(x)$ is in w' for any object x in \mathcal{C} . We let \mathbf{RelCat} (resp. \mathbf{RelCat}^+) denote the 2-category of essentially small relative categories and relative functors and relative natural equivalences (resp. relative categories and relative functors and natural transformations).

(ii) Relative functors $f, f' : \mathbf{C} \rightarrow \mathbf{C}'$ are **weakly homotopic** if there is a zig-zag sequence of relative natural transformations connecting f to f' . A relative functor $f : \mathbf{C} \rightarrow \mathbf{C}'$ is a **homotopy equivalence** if there is a relative functor $g : \mathbf{C}' \rightarrow \mathbf{C}$ such that gf and fg are weakly homotopic to identity functors respectively.

(iii) Let \mathcal{R} be a subcategory of \mathbf{RelEx} . A functor F from \mathcal{R} to a category \mathcal{X} is **(categorical) homotopy invariant** if for any relative functors $f, g : \mathbf{C} \rightarrow \mathbf{C}'$ such that f and g are weakly homotopic, we have the equality $F(f) = F(g)$.

(iv) Let $\mathbf{C} = (\mathcal{C}, w)$ be a relative category with a specific zero object 0 in \mathcal{C} . We say that an object x in \mathcal{C} is **w -trivial** if the canonical morphism $0 \rightarrow x$ is in w . We write \mathcal{C}^w for the full subcategory of w -trivial objects in \mathcal{C} .

(7) Exact categories, Waldhausen categories and algebraic K -theory

(i) Basically, for exact categories, we follow the notations in [Qui73] and for connective K -theory of categories with cofibrations and weak equivalences, we follow the notations in [Wal85] and for non-connective K -theory of exact categories, Frobenius pairs and complicial exact categories with weak equivalences, we follow the notations in [Sch04], [Sch06] and [Sch11]. As a comprehensive reference, please see also [Wei13].

(ii) We denote a cofibration and an admissible monomorphism (resp. an admissible epimorphism) by the arrow “ \rightarrow ” (resp. “ \twoheadrightarrow ”). We sometimes denote a cofibration sequence $x \xrightarrow{i} y \xrightarrow{p} z$ by (i, p) . We write the same letter 0 for a specific zero object of a category with cofibrations. We assume that an exact functor between categories with cofibrations (or exact categories) preserves a specific zero object. We denote the 2-category of essentially small exact categories by \mathbf{ExCat} .

(iii) We call a category with cofibrations and weak equivalences a **Waldhausen category** and a complicial exact category with weak equivalences a **complicial Waldhausen category**.

(iv) For a Waldhausen category $\mathbf{X} = (\mathcal{X}, w)$, we write $K^W(\mathbf{X})$ or $K^W(\mathcal{X}; w)$ for the Waldhausen K -theory of \mathbf{X} . We also write $K(\mathcal{X})$ for $K(\mathcal{X}; i)$. For any exact category \mathcal{E} and any complicial Waldhausen category $\mathbf{C} = (\mathcal{C}, v)$, we write $\mathbb{K}^S(\mathcal{E})$ and $\mathbb{K}^S(\mathbf{C}) = \mathbb{K}^S(\mathcal{C}; v)$ for Schlichting non-connective K -theory of \mathcal{E} and \mathbf{C} respectively.

(v) We say that a functor between exact categories (resp. categories with cofibrations) $f : \mathcal{X} \rightarrow \mathcal{Y}$ **reflects exactness** if for a sequence $x \rightarrow y \rightarrow z$ in \mathcal{X} such that $fx \rightarrow fy \rightarrow fz$ is an admissible exact sequence (resp. a cofibration sequence) in \mathcal{Y} , $x \rightarrow y \rightarrow z$ is an admissible exact sequence (resp. a cofibration sequence) in \mathcal{X} .

(vi) For an exact category \mathcal{E} , we say that its full subcategory \mathcal{F} is an **exact subcategory** (resp. a **strict exact subcategory**) if it is an exact category and the inclusion functor is exact (and reflects exactness).

(vii) Notice that as in [Wal85, p.321, p.327], the concept of subcategories with cofibrations (resp. Waldhausen subcategories) is stronger than that of exact subcategories. Namely we

say that \mathcal{C}' is a subcategory with cofibrations of a category with cofibration \mathcal{C} if a morphism in \mathcal{C}' is a cofibration in \mathcal{C}' if and only if it is a cofibration in \mathcal{C} and the quotient is in \mathcal{C}' (up to isomorphism). That is, the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ is exact and reflects exactness. For example, let \mathcal{E} be a non-semisimple exact category. Then \mathcal{E} with semi-simple exact structure is not a subcategory with cofibrations of \mathcal{E} , but a exact subcategory of \mathcal{E} .

(viii) Let \mathcal{E} be an exact category and \mathcal{F} a full subcategory of \mathcal{E} . We say that \mathcal{F} is **closed under kernels (of admissible epimorphisms)** if for any admissible exact sequence $x \twoheadrightarrow y \twoheadrightarrow z$ in \mathcal{E} if y is isomorphic to object in \mathcal{F} , then x is also isomorphic to an object in \mathcal{F} . (See [Wei13, II.7.0]).

(ix) We say that the class of morphisms w in an exact category \mathcal{E} satisfies the **cogluing axiom** if $(\mathcal{E}^{\text{op}}, w^{\text{op}})$ satisfies the gluing axiom.

(x) A pair of an exact category \mathcal{E} and a class of morphisms w in \mathcal{E} is said to be a **Waldhausen exact category** if (\mathcal{E}, w) and $(\mathcal{E}^{\text{op}}, w^{\text{op}})$ are Waldhausen categories. We let $\mathbf{WalEx}^\#$ denote the 2-subcategory of essentially small Waldhausen exact categories and exact functors in $\mathbf{RelCat}^\#$ for $\# \in \{+, \text{nothing}\}$.

(xi) For a Waldhausen category (\mathcal{C}, w) , we write $w(\mathcal{C})$ if we wish to emphasis that w is the class of weak equivalences in \mathcal{C} . We sometimes write \mathcal{C} for (\mathcal{C}, w) when w is the class of all isomorphisms in \mathcal{C} .

(xii) (cf. [Sch06, p.129 Definition 11]). Let \mathcal{C} be a category with cofibrations and w a class of morphisms in w . We say that w or (\mathcal{C}, w) satisfies the **factorization axiom** (resp. **extensional axiom**) if for any morphism $f : x \rightarrow y$ in \mathcal{C} , there is a cofibration $i : x \rightarrow z$ and a morphism $a : z \rightarrow y$ in w such that $f = ai$ (resp. w is closed under extensions). In this case, moreover if (\mathcal{C}, w) is a relative category or a Waldhausen category, then we say that (\mathcal{C}, w) is a factorization relative category or an extensional Waldhausen category and so on respectively.

(xiii) A morphism of Waldhausen categories $f : (\mathcal{C}, w) \rightarrow (\mathcal{C}', w')$ is a K^W -**equivalence** if it induces a homotopy equivalence on Waldhausen K -theory.

(xiv) Let \mathcal{Z} be a category with cofibrations and \mathcal{X}, \mathcal{Y} subcategories with cofibrations. We write $E(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$ for the category with cofibrations of cofibration sequences $x \twoheadrightarrow z \twoheadrightarrow y$ such that x is in \mathcal{X} and y is in \mathcal{Y} . There are three exact functors $s_{E(\mathcal{X}, \mathcal{Z}, \mathcal{Y})}$, $m_{E(\mathcal{X}, \mathcal{Z}, \mathcal{Y})}$ and $q_{E(\mathcal{X}, \mathcal{Z}, \mathcal{Y})}$ or shortly s , m and q from $E(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$ to \mathcal{X} , \mathcal{Z} and \mathcal{Y} which send a cofibration sequence $x \twoheadrightarrow z \twoheadrightarrow y$ to x , z and y respectively. We let $\text{In}_{\mathcal{X}}^l$ (resp. $\text{In}_{\mathcal{Y}}^r$) denote an exact functor from \mathcal{X} (resp. \mathcal{Y}) to $E(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$ which sends an object a in \mathcal{X} (resp. \mathcal{Y}) to a cofibration sequence $a \xrightarrow{\text{id}_a} a \xrightarrow{0} 0$ (resp. $0 \xrightarrow{0} a \xrightarrow{\text{id}_a} a$). If there is a class of morphisms w in \mathcal{Z} , then we define the class of morphisms $E(w)$ in $E(\mathcal{X}, \mathcal{Z}, \mathcal{Y})$ by $s^{-1}(w) \cap m^{-1}(w) \cap q^{-1}(w)$. We write $E(\mathcal{Z})$ for $E(\mathcal{Z}, \mathcal{Z}, \mathcal{Z})$.

(xv) A sequence of exact functors $f \xrightarrow{A} g \xrightarrow{B} h$ between exact categories $\mathcal{E} \rightarrow \mathcal{E}'$ is admissible exact if for any object x in \mathcal{E} , the sequence $f(x) \xrightarrow{A(x)} g(x) \xrightarrow{B(x)} h(x)$ is an admissible exact sequence in \mathcal{E}' .

(8) Triangulated categories

(i) We follows the notations about triangulated category theory for [Kel96] and [Nee01]. We denote a triangulated category by $(\mathcal{T}, \Sigma, \Delta)$ or simply \mathcal{T} where \mathcal{T} is an additive category, Σ is an additive self category equivalence on \mathcal{T} which is said to be the **suspension functor** and Δ is a class of sequences in \mathcal{T} of the form

$$x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} \Sigma x \quad (1)$$

such that $vu = 0$ and $wv = 0$ which we denote by (u, v, w) and call it a (Σ) -**exact triangle** and they satisfies the usual Verdier axioms from **(TR 1)** to **(TR 4)**. In the sequence (1), we sometimes write $\text{Cone } u$ for the object z . A **triangle functor** between triangulated categories from (\mathcal{T}, σ) to (\mathcal{T}', σ') is a pair (f, α) consisting of an additive functor $f : \mathcal{T} \rightarrow \mathcal{T}'$ and a natural equivalence $\alpha : f\Sigma \rightarrow \Sigma'f$ such that they preserves exact triangles. A **triangle natural trans-**

formation $\theta : (f, \alpha) \rightarrow (g, \beta)$ between triangulated functors $(f, \alpha), (g, \beta) : (\mathcal{T}, \Sigma) \rightarrow (\mathcal{T}', \Sigma')$ is a natural transformation $\theta : f \rightarrow g$ such that $(\Sigma'(\theta))\alpha = \beta(\theta(\Sigma))$. Let us denote the 2-category of essentially small triangulated categories by **TriCat**.

(ii) Let (\mathcal{T}, Σ) be a triangulated category. We say that a full subcategory \mathcal{D} of \mathcal{T} is a **quasi-triangulated subcategory (of \mathcal{T})** if (\mathcal{D}, Σ) is a triangulated category and the inclusion functor $(\iota, \text{id}_\Sigma) : \mathcal{D} \rightarrow \mathcal{T}$ is a triangulated functor. We say that a quasi-triangulated subcategory \mathcal{D} of \mathcal{T} is a **triangulated subcategory (of \mathcal{T})** if it is tight. Namely an additive full subcategory \mathcal{D} of \mathcal{T} is a triangulated subcategory if $\Sigma^\pm(\mathcal{D}) \subset \mathcal{D}$ and if for any Σ -exact triangle $x \rightarrow y \rightarrow z \rightarrow \Sigma x$ in \mathcal{T} , if x and y are in \mathcal{D} , then z is also in \mathcal{D} . Assuming the condition $\Sigma^\pm(\mathcal{D}) \subset \mathcal{D}$, the last condition is equivalent to the condition that if two of x, y and z are in \mathcal{D} , then the other one is also in \mathcal{D} . We call this condition the **two out of three (for Σ -exact triangles) axiom**.

(iii) We say that a triangulated subcategory \mathcal{D} of \mathcal{T} is **thick** if \mathcal{D} is closed under direct summand. Namely for any objects x and y in \mathcal{T} , if $x \oplus y$ is in \mathcal{D} , then both x and y are also in \mathcal{D} .

(iv) Let \mathcal{T} be a triangulated category and \mathcal{D} a full subcategory of \mathcal{T} . We write $\mathcal{D}_{\text{tri}, \mathcal{T}}$ or shortly \mathcal{D}_{tri} (resp. $\mathcal{D}_{\text{thi}, \mathcal{T}}$ or shortly \mathcal{D}_{thi}) for the smallest triangulated subcategory (resp. thick subcategory) contains \mathcal{D} in \mathcal{T} and call it **triangulated** (resp. **thick**) **closure of \mathcal{D} (in \mathcal{T})**.

(v) Let $f : \mathcal{T} \rightarrow \mathcal{T}'$ be a triangulated functor between triangulated categories. The **(triangulated) image of f** $(\text{Im } f)_{\text{tri}}$ or shortly $\text{Im } f$ is the smallest triangulated category which contains the full subcategory spanned by $f(\text{Ob } \mathcal{T})$. The **(triangulated) cokernel of f** $(\text{Coker } f)_{\text{tri}}$ or shortly $\text{Coker } f$ is defined by the Verdier quotient $\text{Coker } f := \mathcal{T}' / \text{Im } f$.

(vi) (cf. [Sch06, Definition 1]). A sequence of triangulated categories $\mathcal{T} \xrightarrow{i} \mathcal{T}' \xrightarrow{p} \mathcal{T}''$ is **exact** (resp. **weakly exact**) if pi is isomorphic to the zero functor, i and p induce equivalence of triangulated categories $\mathcal{T} \xrightarrow{\sim} \text{Ker } p$ and $\mathcal{T}' / \text{Ker } p \xrightarrow{\sim} \mathcal{T}''$. (resp. i is fully faithful and the induced functor $\mathcal{T}' / \mathcal{T} \rightarrow \mathcal{T}''$ is **cofinal**. The last condition means that it is fully faithful, and every object in \mathcal{T}'' is a direct summand of an object of $\mathcal{T}' / \mathcal{T}$.)

(vii) (cf. [Kel96, §8]). Let $(R, \rho) : (\mathcal{S}, \Sigma) \rightarrow (\mathcal{T}, \Sigma')$ and $(L, \lambda) : \mathcal{T} \rightarrow \mathcal{S}$ be two triangle functors such that L is left adjoint to R . Let $A : \text{id}_\mathcal{T} \rightarrow RL$ and $B : LR \rightarrow \text{id}_\mathcal{S}$ be adjunction morphisms. For any objects x in \mathcal{T} and y in \mathcal{S} , we write $\mu(x, y)$ for the bijection $\text{Hom}_\mathcal{S}(Lx, y) \rightarrow \text{Hom}_\mathcal{T}(x, Ry)$, $f \mapsto (Rf)(Ax)$. We say that (L, λ) (resp. (R, ρ)) is **left** (resp. **right**) **triangle adjoint** to (R, ρ) (resp. (L, λ)) if the following equivalent conditions hold.

(a) $\lambda = (B\Sigma L)(L\rho^{-1}L)(L\Sigma'A)$.

(b) $\rho^{-1} = (R\Sigma B)(R\lambda R)(A\Sigma'R)$.

(c) $B\Sigma = (\Sigma B)(\lambda R)(L\rho)$.

(d) $\Sigma'A = (\rho L)(R\lambda)(B\Sigma')$.

(e) $\mu(\Sigma', \Sigma) \text{Hom}(\lambda, \Sigma) \Sigma = \text{Hom}(\Sigma', \rho^{-1}) \Sigma' \mu$. Namely for any objects x in \mathcal{T} and y in \mathcal{S} , the diagram below is commutative.

$$\begin{array}{ccccc}
 \text{Hom}_\mathcal{S}(Lx, y) & \xrightarrow{\Sigma} & \text{Hom}_\mathcal{S}(\Sigma Lx, \Sigma y) & \xrightarrow{\text{Hom}(\lambda x, \Sigma y)} & \text{Hom}_\mathcal{S}(L\Sigma'x, \Sigma y) \\
 \mu(x, y) \downarrow & & & & \downarrow \mu(\Sigma'x, \Sigma y) \\
 \text{Hom}_\mathcal{T}(x, Ry) & \xrightarrow{\Sigma'} & \text{Hom}_\mathcal{T}(\Sigma x, \Sigma' Ry) & \xrightarrow{\text{Hom}(\Sigma'x, R\Sigma y)} & \text{Hom}_\mathcal{T}(\Sigma'x, R\Sigma y).
 \end{array}$$

(9) Chain complexes

(i) For the notations about chain complexes, we basically follow in [Wei94]. For a chain complex, we use the homological notation. Namely a boundary morphisms are degree -1 . For an additive category \mathcal{B} , we denote the category of bounded complexes on \mathcal{B} by $\text{Ch}_b(\mathcal{B})$. There exists the canonical functor $j_\mathcal{B} : \mathcal{B} \rightarrow \text{Ch}_b(\mathcal{B})$ where $j_\mathcal{B}(x)_k$ is x if $k = 0$ and 0 if $k \neq 0$.

(ii) Let $f : x \rightarrow y$ be a morphism between complexes in $\text{Ch}_b(\mathcal{B})$ and k an integer. We define the complex $x[k]$ and morphism $f[k] : x[k] \rightarrow y[k]$ by $x[k]_n = x_{n+k}$ and $d_n^{x[k]} = (-1)^k d_{n+k}^x$ and $f[k]_n = f_{n+k}$.

(iii) For a complex x in an additive category \mathcal{B} , we define the brutal truncation $\sigma_{\geq k}x$ as follows. $(\sigma_{\geq k}x)_i$ is equal to 0 if $i < k$ and x_i if $i \geq k$ and we put $\sigma_{\leq k-1}x := x/\sigma_{\geq k}x$. If $\text{Ker } d_{k+1}^x$ (resp. $\text{Im } d_{k+1}^x$) exists, then we put the complex $\tau_{\leq k}x$ (resp. $\tau_{\geq k+1}x$) as follows. $(\tau_{\leq k}x)_n$ (resp. $(\tau_{\geq k+1}x)_n$) is x_n if $n \leq k$ (resp. $n \geq k+2$), is $\text{Ker } d_{k+1}^x$ (resp. $\text{Im } d_{k+1}^x$) if $n = k+1$ and 0 for otherwise.

(iv) For any chain morphism $f : x \rightarrow y$ of complexes in an additive category \mathcal{B} , we define the **mapping cone** of f , $\text{Cone } f$ by the formula $(\text{Cone } f)_n = x_{n-1} \oplus y_n$ and $d_n^{\text{Cone } f} = \begin{pmatrix} -d_{n-1}^x & 0 \\ -f_{n-1} & d_n^y \end{pmatrix}$.

(v) For a pair of integers $a \leq b$, let $\mathbf{Ch}_{[a,b]}(\mathcal{B})$ be the full subcategory of $\mathbf{Ch}_b(\mathcal{B})$ consisting of those complexes z such that $z_n = 0$ unless $n \in [a, b]$. For any z in $\mathbf{Ch}_b(\mathcal{B})$, we put

$$\text{length } z := \min\{b - a; z \in \mathbf{Ch}_{[a,b]}(\mathcal{B})\}.$$

and call it the **length** of z .

(vi) We denote the homotopy category of $\mathbf{Ch}_b(\mathcal{B})$ by $\mathcal{H}_b(\mathcal{B})$. Namely the class of objects of $\mathcal{H}_b(\mathcal{B})$ is same as $\mathbf{Ch}_b(\mathcal{B})$, morphisms of $\mathcal{H}_b(\mathcal{B})$ is chain homotopy classes of morphisms in $\mathbf{Ch}_b(\mathcal{B})$ and the composition of morphisms is induced from $\mathbf{Ch}_b(\mathcal{B})$. Therefore there exists the canonical additive functor $P_{\mathcal{B}} : \mathbf{Ch}_b(\mathcal{B}) \rightarrow \mathcal{H}_b(\mathcal{B})$. It is well-known that $\mathcal{H}_b(\mathcal{B})$ naturally becomes a triangulated category. (cf. [Kel96, 6.2, §7]).

(vii) A **strictly acyclic complex** in a Quillen exact category is a chain complex which decomposed into admissible short exact sequences (see [Kel96, §11]). **Acyclic complexes** are chain complexes which are chain homotopy equivalent to strictly acyclic complexes. A morphism between chain complexes is a **quasi-isomorphism** if its mapping cone is an acyclic complex. We denote the category of bounded acyclic complexes on a Quillen exact category \mathcal{E} by $\text{Acy}_b(\mathcal{E})$. It is well-known that $P_{\mathcal{E}}(\text{Acy}_b(\mathcal{E}))$ the full subcategory of $\mathcal{H}_b(\mathcal{E})$ spanned by $P_{\mathcal{E}}(\text{Ob } \text{Acy}_b(\mathcal{E}))$ where $P_{\mathcal{E}}$ is the canonical projection functor $P_{\mathcal{E}} : \mathbf{Ch}_b(\mathcal{E}) \rightarrow \mathcal{H}_b(\mathcal{E})$, is a triangulated subcategory. (cf. [Kel96, 11.3]). We define the **bounded derived category** of \mathcal{E} , $\mathcal{D}_b(\mathcal{E})$ by $\mathcal{D}_b(\mathcal{E}) := \mathcal{H}_b(\mathcal{B})/P_{\mathcal{E}}(\text{Acy}_b(\mathcal{E}))$.

1 Triangulated subcategories

In this section, we will study the lattices structure and the functorial behaviour of the set of (thick) triangulated subcategories of triangulated categories. The key proposition 1.5 says that there exists the canonical lattice isomorphism between a lattice of (thick) triangulated subcategories of a triangulated category and that of the quotient triangulated category. Utilizing this proposition and the notion of *factorizable pairs*, we explicitly describe the join of a factorizable pair of thick subcategories in 1.16 and the join of general triangulated subcategories in 1.17. Recall the conventions of partially ordered sets and triangulated categories from Conventions (3) and (8). We start by introducing useful lemmata 1.1 and 1.2 to study the lattices of triangulated subcategories.

Lemma 1.1. *Let L be a \vee -complete (resp. \wedge -complete) lattice with the minimum (resp. maximum) element. Then L is a complete lattice.*

Proof. We only prove for a \vee -complete lattice L . To prove for a \wedge -complete lattice is similar. Let $S \subset L$ be a non-empty subset. Then we put

$$l(S) := \{u \in L; u \geq x \text{ for any } x \in S\}.$$

Since the minimum element is in $l(S)$, $l(S)$ is not the empty set. Therefore there exists the element $\inf S = \sup l(S)$ in L . \square

Lemma 1.2. *For objects x and y in a triangulated category \mathcal{T} , if x is a retraction of y , then x is a direct summand of y . In particular, a triangulated subcategory \mathcal{D} of \mathcal{T} is thick if and only if it closed under retractions.*

Proof. Let $i : x \rightarrow y$ and $p : y \rightarrow x$ be morphisms such that $pi = \text{id}_x$. Let us consider the diagram of Σ -exact triangles below

$$\begin{array}{ccccccc}
 x & \xrightarrow{i} & y & \xrightarrow{q} & z & \xrightarrow{r} & \Sigma x \\
 \text{id}_x \downarrow & & \begin{pmatrix} p \\ q \end{pmatrix} \downarrow & & \downarrow h & & \downarrow \text{id}_x \\
 x & \xrightarrow{\begin{pmatrix} \text{id}_x \\ 0 \end{pmatrix}} & x \oplus z & \xrightarrow{\begin{pmatrix} 0 & \text{id}_z \end{pmatrix}} & z & \xrightarrow{0} & \Sigma x.
 \end{array}$$

By the axiom of triangulated categories, there exists a morphism h which makes diagram above commutative. Therefore it turns out that $r = 0$. Hence we can take $h = \text{id}_z$. Now by the five lemma of Σ -exact triangles, we learn that $\begin{pmatrix} p \\ q \end{pmatrix} : y \rightarrow x \oplus z$ is an isomorphism. \square

Definition 1.3. Let \mathcal{T} be an essentially small triangulated category and \mathcal{L} a triangulated subcategory of \mathcal{T} . We write $V_\emptyset(\mathcal{L})$ or $V(\mathcal{L})$ (resp. $V_{\text{thi}}(\mathcal{L})$) for the set of all (thick) triangulated subcategories which contains \mathcal{L} . In particular, we put $\text{Tri}_\#(\mathcal{T}) := V_\#(\{0\})$ where $\# = \emptyset$ or thi . Obviously $V_\#(\mathcal{L})$ is a partially ordered set with the usual inclusion order.

Lemma-Definition 1.4. Let \mathcal{T} and \mathcal{T}' be essentially small triangulated categories and $f : \mathcal{T} \rightarrow \mathcal{T}'$ a triangulated functor. Then

- (1) For any triangulated subcategory \mathcal{N} of \mathcal{T} , $V_\#(\mathcal{N})$ ($\# = \emptyset$ or thi) is a complete lattice. We denote the join and the meet in $V_\#(\mathcal{N})$ by $\vee_\#$ and $\wedge_\#$ respectively.
- (2) For any $\mathcal{N} \in \text{Tri}_\#(\mathcal{T}')$, $f^{-1}\mathcal{N}$ is in $\text{Tri}_\#(\mathcal{T})$.
- (3) The association $\text{Tri}_\#(f) : \text{Tri}_\#(\mathcal{T}') \rightarrow \text{Tri}_\#(\mathcal{T})$, $\mathcal{N} \mapsto f^{-1}\mathcal{N}$ is an order preserving map. In particular, there exist the functors

$$\text{Tri}_\# : \mathbf{TriCat}^{\text{op}} \rightarrow \mathbf{CLat} \quad (\# = \emptyset \text{ or } \text{Tri}).$$

Proof. Obviously $V_\#(\mathcal{N})$ has the maximum element \mathcal{T} and is $\wedge_\#$ -complete. (The operation $\wedge_\#$ is just the intersection). Therefore by 1.1, it is complete. Assertions (2) and (3) are straightforwards. \square

In this section, from now on, let \mathcal{T} be a triangulated category.

Proposition 1.5. *For a thick subcategory \mathcal{L} of \mathcal{T} , the canonical projection $\pi : \mathcal{T} \rightarrow \mathcal{T} / \mathcal{L}$ induces the isomorphism*

$$\text{Tri}_\#(\pi) : \text{Tri}_\#(\mathcal{T} / \mathcal{L}) \xrightarrow{\sim} V_\#(\mathcal{L}), \quad \mathcal{N} \mapsto \pi^{-1}\mathcal{N}$$

where $\# = \emptyset$ or thi .

Proof. The proof is carried out in several steps.

1.6 (Step 1). We will construct the inverse map of $\text{Tri}_\#(\pi)$. To do so, we need to prove that for any (thick) triangulated subcategory \mathcal{N} in $V_\#(\mathcal{L})$, $\pi(\mathcal{N})$ the isomorphisms closure of the full subcategory spanned by $\pi(\text{Ob } \mathcal{N})$ is a (thick) triangulated subcategory. Then the association $\mathcal{N} \mapsto \pi(\mathcal{N})$ is the desired inverse map.

1.7 (Step 2). First we will prove that for any object x in $\pi(\mathcal{N})$, Σx and $\Sigma^{-1}x$ are in $\pi(\mathcal{N})$ again. By definition, there exists an object y in \mathcal{N} , x is isomorphic to πy in \mathcal{T}/\mathcal{L} . Therefore we have the isomorphisms

$$\Sigma^{\pm 1}x \xrightarrow{\sim} \Sigma^{\pm 1}\pi y \xrightarrow{\sim} \pi \Sigma^{\pm 1}y$$

with $\Sigma^{\pm 1}y \in \mathcal{N}$. Hence we get the assertion.

1.8 (Step 3). Next we prove that for any Σ -exact triangle

$$x \xrightarrow{u} y \rightarrow z \rightarrow \Sigma x$$

in \mathcal{T}/\mathcal{L} , if x and y are in $\pi(\mathcal{N})$, then z is also in $\pi(\mathcal{N})$. We represent the corresponding objects of x and y in \mathcal{T} by the same letters x and y respectively and shall assume that both x and y are in \mathcal{N} . The morphism u is represented by a morphisms $x \xleftarrow{s} w \xrightarrow{f_u} y$ in \mathcal{T} with $\text{Cone } s_u \in \mathcal{L}$. Since \mathcal{L} is contained in \mathcal{N} , by the two out of three axiom, we learn that w is in \mathcal{N} and therefore by the two out of three axiom again, $\text{Cone } f_u$ is also in \mathcal{N} . Since z is isomorphic to $\pi(\text{Cone } f_u)$, we notice that z is in $\pi(\mathcal{N})$.

1.9 (Step 4). We will prove that for any object x in \mathcal{T} , $\pi(x)$ is in $\pi(\mathcal{N})$ if and only if x is in \mathcal{N} . Let us assume that $\pi(x)$ is in $\pi(\mathcal{N})$. Then there exists an object y in \mathcal{N} such that $\pi(y)$ is isomorphic to $\pi(x)$. The isomorphism between $\pi(x)$ and $\pi(y)$ in \mathcal{T}/\mathcal{L} is represented by morphisms $x \xleftarrow{s} z \xrightarrow{t} y$ with $\text{Cone } s, \text{Cone } t \in \mathcal{L} (\subset \mathcal{N})$. Therefore by the two out of three axiom, z and x are in \mathcal{N} . The converse assertion is trivial.

1.10 (Step 5). We prove that if \mathcal{N} is thick, then $\pi(\mathcal{N})$ is also thick. We just check that for any objects x and y in \mathcal{T} if $\pi(x) \oplus \pi(y)$ is in $\pi(\mathcal{N})$, then $\pi(x)$ and $\pi(y)$ are in $\pi(\mathcal{N})$. By 1.9, $x \oplus y$ is in \mathcal{N} . Therefore we get the assertion.

1.11 (Step 6). We prove $\text{Tri}_{\#}(\pi)$ and π are inverse functors in each other. First we prove that for any \mathcal{N} in $\text{Tri}_{\#}(\mathcal{T})$, we have the equality $\text{Tri}_{\#}(\pi)(\pi(\mathcal{N})) = \mathcal{N}$. For any object x in $\text{Tri}_{\#}(\pi)(\pi(\mathcal{N}))$, $\pi(x)$ is in $\pi(\mathcal{N})$ and therefore by 1.9, x is in \mathcal{N} . Hence we get the result. Finally we will prove that for any \mathcal{N}' in $\text{Tri}_{\#}(\mathcal{T}/\mathcal{L})$, we have the equality $\pi(\text{Tri}_{\#}(\pi)(\mathcal{N}')) = \mathcal{N}'$. For any object y in \mathcal{T} such that $\pi(y)$ is in $\pi(\text{Tri}_{\#}(\pi)(\mathcal{N}'))$, by 1.9, y is in $\text{Tri}_{\#}(\pi)(\mathcal{N}')$ and this is equivalent to the condition that $\pi(y)$ is in \mathcal{N}' .

□

Definition 1.12 (Factorizable pair). Let \mathcal{M} and \mathcal{N} be triangulated subcategories of \mathcal{T} . We say that the ordered pair $(\mathcal{N}, \mathcal{M})$ is **factorizable (in \mathcal{T})** if any morphism from an object x in \mathcal{N} to an object y in \mathcal{M} admits a factorization $x \rightarrow z \rightarrow y$ with $z \in \mathcal{N} \cap \mathcal{M}$.

Lemma 1.13 (Quotient of factorizable pairs). Let $(\mathcal{N}, \mathcal{M})$ be a factorizable pair in \mathcal{T} , then $((\mathcal{N}/\mathcal{N} \cap \mathcal{M})_{\text{isom}}, (\mathcal{M}/\mathcal{N} \cap \mathcal{M})_{\text{isom}})$ is factorizable in $\mathcal{T}/\mathcal{N} \cap \mathcal{M}$. Namely any morphism in $\mathcal{T}/\mathcal{N} \cap \mathcal{M}$ from an object in $\mathcal{N}/\mathcal{N} \cap \mathcal{M}$ to an object in $\mathcal{M}/\mathcal{N} \cap \mathcal{M}$ is the zero morphism.

Proof. First notice that by 1.5, $(\mathcal{N}/\mathcal{N} \cap \mathcal{M})_{\text{isom}}$ and $(\mathcal{M}/\mathcal{N} \cap \mathcal{M})_{\text{isom}}$ are triangulated subcategories of $\mathcal{T}/\mathcal{N} \cap \mathcal{M}$. For any morphism from an object x in $\mathcal{N}/\mathcal{N} \cap \mathcal{M}$ to an object y in $\mathcal{M}/\mathcal{N} \cap \mathcal{M}$ is written by $x \xleftarrow{s} z \xrightarrow{f} y$ with $\text{Cone } s \in \mathcal{N} \cap \mathcal{M}$. Then we notice that z is in \mathcal{M} and therefore $z \xrightarrow{f} y$ admits a factorization $z \rightarrow w \rightarrow y$ with $w \in \mathcal{N} \cap \mathcal{M}$. □

Proposition 1.14. Let \mathcal{M} and \mathcal{N} be triangulated subcategories of \mathcal{T} . Let us assume that any morphism from an object in \mathcal{M} to an object in \mathcal{N} is the zero morphism. Then

(1) The composition

$$\mathcal{M} \hookrightarrow \mathcal{T} \xrightarrow{Q} \mathcal{T}/\mathcal{N}$$

is fully faithful where Q is the canonical quotient functor.

(2) $\mathcal{M}_{\text{isom}, \mathcal{T}/\mathcal{N}}$ is a triangulated subcategory in \mathcal{T}/\mathcal{N} .

(3) Moreover if \mathcal{M} is a thick subcategory in \mathcal{T} , then $\mathcal{M}_{\text{isom}, \mathcal{T}/\mathcal{N}}$ is a thick subcategory in \mathcal{T}/\mathcal{N} .

Proof. (1) Let x and y be objects in \mathcal{M} . We prove that any morphism in $\text{Hom}_{\mathcal{T}/\mathcal{N}}(x, y)$ is represented by a morphism in $\text{Hom}_{\mathcal{T}}(x, y)$ uniquely. Let $x \xleftarrow{s} z \xrightarrow{f} y$ be a morphism from x to y in \mathcal{T}/\mathcal{N} . Then since $\text{Cone } s$ is in \mathcal{N} , by the assumption, s is an isomorphism. This means that we have the equality $(x \xleftarrow{s} z \xrightarrow{f} y) = (x \xleftarrow{\text{id}_x} z \xrightarrow{fs^{-1}} y)$.

(2) First we prove that $\mathcal{M}_{\text{isom}, \mathcal{T}/\mathcal{N}}$ is closed under Σ^\pm . For any object x in $\mathcal{M}_{\text{isom}, \mathcal{T}/\mathcal{N}}$, there are an object y in \mathcal{M} and an isomorphism $x \xrightarrow{\sim} Q(y)$ in \mathcal{T}/\mathcal{N} . Then we have the isomorphisms

$$\Sigma^\pm x \xrightarrow{\sim} \Sigma^\pm Qy \xrightarrow{\sim} Q\Sigma^\pm y.$$

Since $\Sigma^\pm y$ is in \mathcal{M} , we learn that $\Sigma^\pm x$ is in $\mathcal{M}_{\text{isom}, \mathcal{T}/\mathcal{N}}$. Second we prove that for any Σ -exact triangle $x \rightarrow y \rightarrow z \rightarrow \Sigma x$ in \mathcal{T}/\mathcal{N} , if x and y are in $\mathcal{M}_{\text{isom}, \mathcal{T}/\mathcal{N}}$, then z is also in $\mathcal{M}_{\text{isom}, \mathcal{T}/\mathcal{N}}$. Then there are objects x' and y' in \mathcal{M} such that $Q(x') \xrightarrow{\sim} x$ and $Q(y') \xrightarrow{\sim} y$ in \mathcal{T}/\mathcal{N} . Then by (1), there exists the morphism $u : x' \xleftarrow{s_u} z' \xrightarrow{f_u} y'$ in \mathcal{T}/\mathcal{N} which makes the diagram below commutative.

$$\begin{array}{ccccccc} x & \longrightarrow & y & \longrightarrow & z & \longrightarrow & \Sigma x \\ \wr \downarrow & & \wr \downarrow & & \wr \downarrow & & \wr \downarrow \\ Q(x') & \xrightarrow{u} & Q(y') & \longrightarrow & Q(\text{Cone } f_u) & \longrightarrow & Q(x'). \end{array}$$

Therefore we have an isomorphism $Q(\text{Cone } f_u)$ and z , and it turns out that z is in $\mathcal{M}_{\text{isom}, \mathcal{T}/\mathcal{N}}$.

(3) Let x and y be objects in \mathcal{T} and let us assume that $Q(x \oplus y)$ is in $\mathcal{M}_{\text{isom}, \mathcal{T}/\mathcal{N}}$. We will prove that $Q(x)$ is in $\mathcal{M}_{\text{isom}, \mathcal{T}/\mathcal{N}}$. Then there are an object u in \mathcal{M} and an isomorphism $Q(x \oplus y) \xleftarrow{a} Q(u) \xrightarrow{b} Q(y)$ in \mathcal{T}/\mathcal{N} . Since $\text{Cone } b$ is in \mathcal{N} , b is an isomorphism. Therefore there exists the morphism $c = ab^{-1} : u \rightarrow x \oplus y$ with $\text{Cone } c \in \mathcal{N}$. Hence by the 3×3 -lemma for Σ -exact triangles, there exists a morphism $z := \Sigma^{-1} \text{Cone}(\text{Pr}_y c) \xrightarrow{d} x$ which makes the diagram below commutative

$$\begin{array}{ccccccc} z & \longrightarrow & u & \longrightarrow & y & \longrightarrow & \text{Cone}(\text{Pr}_y c) \\ d \downarrow & & c \downarrow & & \text{id}_y \downarrow & & \downarrow \\ x & \xrightarrow{i_x} & x \oplus y & \xrightarrow{\text{Pr}_y} & y & \longrightarrow & \Sigma x \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Cone } d & \longrightarrow & \text{Cone } c & \longrightarrow & 0 & \longrightarrow & \Sigma \text{Cone } d \end{array}$$

where the morphisms $i_x : x \rightarrow x \oplus y$ and $\text{Pr}_y : x \oplus y \rightarrow y$ are the canonical morphisms and all horizontal lines are Σ -exact triangles. Then it turns out that $\text{Cone } d$ is isomorphic to $\text{Cone } c$ and hence it is in \mathcal{N} . Hence we have an isomorphism $Q(z) \xrightarrow{\sim} Q(x)$ in \mathcal{T}/\mathcal{N} . Since \mathcal{M} is thick, y is in \mathcal{M} and therefore z is also in \mathcal{M} and it turns out that $Q(x)$ is in $\mathcal{M}_{\text{isom}, \mathcal{T}/\mathcal{N}}$. \square

Proposition 1.15. Let \mathcal{N} and \mathcal{M} be triangulated subcategories of \mathcal{T} . Let us assume that $(\mathcal{M}, \mathcal{N})$ or $(\mathcal{N}, \mathcal{M})$ is factorizable in \mathcal{T} . Then

(1) The canonical functors

$$\mathcal{M}/\mathcal{N} \cap \mathcal{M} \rightarrow \mathcal{T}/\mathcal{N} \text{ and } \mathcal{N}/\mathcal{N} \cap \mathcal{M} \rightarrow \mathcal{T}/\mathcal{M}$$

are fully faithful.

(2) Moreover if \mathcal{N} and \mathcal{M} are thick, then $(\mathcal{M}/\mathcal{N} \cap \mathcal{M})_{\text{isom}, \mathcal{T}/\mathcal{N}}$ (resp. $(\mathcal{N}/\mathcal{N} \cap \mathcal{M})_{\text{isom}, \mathcal{T}/\mathcal{M}}$) is a thick subcategory of \mathcal{T}/\mathcal{N} (resp. \mathcal{T}/\mathcal{M}).

Proof. First notice that if $(\mathcal{N}, \mathcal{M})$ is factorizable in \mathcal{T} , then $(\mathcal{M}^{\text{op}}, \mathcal{N}^{\text{op}})$ is factorizable in \mathcal{T}^{op} . Therefore we shall just check that if $(\mathcal{N}, \mathcal{M})$ is factorizable, then the canonical functor $\mathcal{M}/\mathcal{N} \cap \mathcal{M} \rightarrow \mathcal{T}/\mathcal{N}$ is fully faithful and moreover if both \mathcal{N} and \mathcal{M} are thick, then $(\mathcal{M}/\mathcal{N} \cap \mathcal{M})_{\text{isom}, \mathcal{T}/\mathcal{N}}$ is a thick subcategory of \mathcal{T}/\mathcal{N} . The first assertion is mentioned in [Kel96, 10.3]. To prove the second assertion, let us consider the factorization

$$(\mathcal{M}/\mathcal{N} \cap \mathcal{M})_{\text{isom}, \mathcal{T}/\mathcal{N}} \rightarrow (\mathcal{T}/\mathcal{N} \cap \mathcal{M})_{\text{isom}, \mathcal{T}/\mathcal{N}} \xrightarrow{Q} \mathcal{T}/\mathcal{N}$$

where Q is the canonical quotient functor. $(\mathcal{M}/\mathcal{N} \cap \mathcal{M})_{\text{isom}}$ and $(\mathcal{N}/\mathcal{N} \cap \mathcal{M})_{\text{isom}}$ are thick subcategories of $\mathcal{T}/\mathcal{N} \cap \mathcal{M}$ by 1.5 and the ordered pair $((\mathcal{N}/\mathcal{N} \cap \mathcal{M})_{\text{isom}}, (\mathcal{M}/\mathcal{N} \cap \mathcal{M})_{\text{isom}})$ is factorizable by 1.13. Therefore the assertion follows from 1.14. \square

Corollary 1.16 (\vee_{thi} of $\text{Tri}_{\text{thi}}(\mathcal{T})$). *Let \mathcal{N} and \mathcal{M} be thick subcategories of \mathcal{T} . If $(\mathcal{N}, \mathcal{M})$ or $(\mathcal{M}, \mathcal{N})$ is factorizable in \mathcal{T} , then we have the equalities*

$$\mathcal{M} \vee_{\text{thi}} \mathcal{N} = Q^{-1}((\mathcal{M}/\mathcal{N} \cap \mathcal{M})_{\text{isom}, \mathcal{T}/\mathcal{N}}) = Q'^{-1}((\mathcal{N}/\mathcal{N} \cap \mathcal{M})_{\text{isom}, \mathcal{T}/\mathcal{M}})$$

where $Q : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{N}$ and $Q' : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{M}$ are the canonical quotient functors.

Proof. By the symmetry of \mathcal{M} and \mathcal{N} , we shall just check the first equality. For simplicity, we put $\mathcal{O} := Q^{-1}((\mathcal{M}/\mathcal{N} \cap \mathcal{M})_{\text{isom}, \mathcal{T}/\mathcal{N}})$. Obviously \mathcal{O} contains \mathcal{M} and \mathcal{N} . For any thick subcategory \mathcal{L} which contains \mathcal{M} and \mathcal{N} , we have the fully faithful embeddings

$$\mathcal{M}/\mathcal{N} \cap \mathcal{M} \hookrightarrow \mathcal{L}/\mathcal{N} \hookrightarrow \mathcal{T}/\mathcal{N}$$

by 1.15. Therefore by 1.5, $\mathcal{L} = Q^{-1}((\mathcal{L}/\mathcal{N})_{\text{isom}, \mathcal{T}/\mathcal{N}})$ contains \mathcal{O} . Hence we have $\mathcal{M} \vee_{\text{thi}} \mathcal{N} = \mathcal{O}$. \square

For general \mathcal{N} and \mathcal{M} in the proposition above, we need more subtle argument.

Proposition 1.17 ($\vee_{\#}$ of $\text{Tri}_{\#}(\mathcal{T})$ II). *Let \mathcal{N} and \mathcal{M} be triangulated subcategories of \mathcal{T} and $Q : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{N}$ the canonical quotient functor. Then*

(1) *By abuse of the notations, we write $\mathcal{N} \vee_{\text{thi}} \mathcal{M}$ for the smallest thick subcategory of \mathcal{T} which contains both \mathcal{N} and \mathcal{M} . Then we have*

$$\mathcal{N} \vee_{\text{thi}} \mathcal{M} = Q^{-1}(\text{Im}(\mathcal{M} \rightarrow \mathcal{T} \xrightarrow{Q} \mathcal{T}/\mathcal{N}))_{\text{thi}}.$$

In particular, we have the formula

$$\text{Coker}(\mathcal{N}/\mathcal{N} \cap \mathcal{M} \rightarrow \mathcal{T}/\mathcal{M}) \xrightarrow{\sim} \mathcal{T}/(\mathcal{N} \vee_{\text{thi}} \mathcal{M}).$$

(2) *If \mathcal{N} is thick, then we have*

$$\mathcal{N} \vee \mathcal{M} = Q^{-1}(\text{Im}(\mathcal{M} \rightarrow \mathcal{T} \xrightarrow{Q} \mathcal{T}/\mathcal{N})).$$

Proof. For simplicity, we put $\# = \emptyset$ if \mathcal{N} is thick and $\# = \text{thi}$ if \mathcal{N} is not thick. Moreover let us put $\mathcal{Y} = \text{Im}(\mathcal{M} \rightarrow \mathcal{T} \xrightarrow{Q} \mathcal{T}/\mathcal{N})_{\#}$ and $\mathcal{X} = Q^{-1}\mathcal{Y}$. Since \mathcal{Y} is in $\text{Tri}_{\#}(\mathcal{N}_{\#})$, by 1.5, $\mathcal{X} = Q^{-1}\mathcal{Y}$ is in $V_{\#}(\mathcal{N}_{\#})$. Since $Q(x)$ is in \mathcal{Y} for any object x in \mathcal{M} , \mathcal{X} is also in $V_{\#}(\mathcal{M})$. Next let us take $\mathcal{L} \in V_{\#}(\mathcal{N}) \cap V_{\#}(\mathcal{M})$ and put $\mathcal{Z} = \text{Im}(\mathcal{L} \rightarrow \mathcal{T} \xrightarrow{Q} \mathcal{T}/\mathcal{N})_{\#}$. Then obviously we have $\mathcal{Y} \subset \mathcal{Z}$, therefore $\mathcal{X} = Q^{-1}\mathcal{Y} \subset Q^{-1}\mathcal{Z} = \mathcal{L}$. Here the last equality follows from 1.5. \square

2 Relative exact categories

In this section, we study relative exact categories. In particular, we define the bounded derived categories of strict relative exact categories. We start by preparing a useful lemma to treat relative exact categories.

Lemma 2.1. *Let \mathcal{C} be a category with cofibrations and w a class of morphisms in \mathcal{C} such that w contains all identity morphisms in \mathcal{C} . Then*

- (1) *If w satisfies either the extensional or the gluing axiom, then w contains all isomorphisms in \mathcal{C} .*
- (2) *If w satisfies the extensional axiom, then w is closed under co-base change along cofibrations.*

Proof. Assertion (2) is proven in [Moc10, A.21]. We will give a proof of (1). We denote the zero object in \mathcal{C} by 0 . Let $f : x \rightarrow y$ be an isomorphism in \mathcal{C} . Then $f = \text{id}_x \sqcup_{\text{id}_0} \text{id}_0$ by the push-out diagram below and f is also extension of id_x and id_0 by the commutative diagram of admissible short exact sequences below.

$$\begin{array}{ccccc}
 & & 0 & \xrightarrow{\quad} & 0 \\
 & \swarrow & \downarrow \text{id}_x & \swarrow & \downarrow \text{id}_0 \\
 x & \xrightarrow{\quad} & x & & \\
 \downarrow \text{id}_x & & \downarrow & & \downarrow \\
 & & 0 & \xrightarrow{\quad} & 0 \\
 & \swarrow & \downarrow f & \swarrow & \downarrow \\
 x & \xrightarrow{\quad} & y & & \\
 & \searrow & & \searrow & \\
 & & 0 & \xrightarrow{\quad} & 0
 \end{array}
 \quad
 \begin{array}{ccccc}
 & & \text{id}_x & & \\
 x & \xrightarrow{\quad} & x & \twoheadrightarrow & 0 \\
 \downarrow \text{id}_x & & \downarrow f & & \downarrow \text{id}_0 \\
 x & \xrightarrow{\quad} & y & \twoheadrightarrow & 0
 \end{array}$$

Therefore if w satisfies either the extensional or the gluing axioms, then f is in w . \square

Recall the notations of relative categories, exact categories and chain complexes from Conventions (6), (7) and (9).

Definition 2.2 (Relative exact categories). (1) A relative category $\mathbf{E} = (\mathcal{E}, w)$ is a **relative exact category** if the underlying category \mathcal{E} is a Quillen exact category with a specific zero object 0 .

(2) A **Relative exact functor** $f : \mathbf{E} = (\mathcal{E}, w) \rightarrow \mathbf{E}' = (\mathcal{E}', w')$ between relative exact categories is a relative functor such that $f : \mathcal{E} \rightarrow \mathcal{E}'$ is an exact functor and $f(0) = 0$.

(3) We denote the 2-subcategory of relative exact categories and relative exact functors in $\mathbf{RelCat}^\#$ by $\mathbf{RelEx}^\#$ for $\# \in \{+, \text{nothing}\}$.

(4) A relative exact functor $f : \mathbf{E} \rightarrow \mathbf{E}'$ is an **exact homotopy equivalence** if there is a relative exact functor $g : \mathbf{E}' \rightarrow \mathbf{E}$ such that both fg and gf are weakly homotopic to identity functors respectively.

(5) A relative exact category $\mathbf{E} = (\mathcal{E}, w)$ is **strict** if \mathcal{E}^w is a strict exact subcategory of \mathcal{E} . A strict relative exact category $\mathbf{E} = (\mathcal{E}, w)$ is **very strict** if the inclusion functor $\mathcal{E}^w \hookrightarrow \mathcal{E}$ induces a fully faithful functor on the bounded derived categories $\mathcal{D}_b(\mathcal{E}^w) \hookrightarrow \mathcal{D}_b(\mathcal{E})$. We denote the full 2-subcategory of strict (resp. very strict) relative exact categories in $\mathbf{RelEx}^\#$ by $\mathbf{RelEx}_{\text{strict}}^\#$ (resp. $\mathbf{RelEx}_{v.s}^\#$) for $\# \in \{+, \text{nothing}\}$.

Remark 2.3. (1) Let $\mathbf{E} = (\mathcal{E}, w)$ be a relative exact category. If w contains all isomorphisms between zero objects in \mathcal{E} , then \mathcal{E}^w does not depend upon a choice of a specific zero object in \mathcal{E} .

(2) A relative exact functor $f : \mathbf{E} = (\mathcal{E}, w) \rightarrow \mathbf{E}' = (\mathcal{E}', w')$ induces a functor $f : \mathcal{E}^w \rightarrow \mathcal{E}'^{w'}$. If both \mathbf{E} and \mathbf{E}' are strict, then the induced functor $f : \mathcal{E}^w \rightarrow \mathcal{E}'^{w'}$ is exact.

Recall the definition of Waldhausen exact categories from Conventions (7) (x).

Proposition 2.4 (Examples of strict relative exact categories). *For any relative exact category $\mathbf{E} = (\mathcal{E}, w)$ if either w satisfies the extensional axiom or \mathbf{E} is a Waldhausen exact category, then \mathbf{E} is a strict relative exact category.*

Proof. In the commutative diagram of admissible short exact sequences in \mathcal{E} below, we have the equalities $a = \text{id}_0 \times_c b$ and $c = \text{id}_0 \sqcup_a b$.

$$\begin{array}{ccccc} x & \xrightarrow{\quad} & y & \twoheadrightarrow & z \\ a \downarrow & & b \downarrow & & \downarrow c \\ 0 & \xrightarrow{\quad} & 0 & \twoheadrightarrow & 0. \end{array}$$

Therefore if w satisfies the extensional (resp. gluing, cogluing) axiom and if x and z (resp. x and y , y and z) are in \mathcal{E}^w , then y (resp. z , x) is also in \mathcal{E}^w . Hence if w satisfies the extensional axiom (resp. \mathbf{E} is a Waldhausen exact category), then \mathcal{E}^w is closed under extensions (resp. taking admissible sub- and quotient objects and finite direct sums) in \mathcal{E} and \mathcal{E}^w is a strict exact subcategory of \mathcal{E} (by [Moc13, 5.3]). \square

In the rest of this section, let $\mathbf{E} = (\mathcal{E}, w)$ and $\mathbf{F} = (\mathcal{F}, v)$ be relative exact categories.

Definition 2.5 (Level and quasi-weak equivalences). (1) A morphism $f : x \rightarrow y$ in $\mathbf{Ch}_b(\mathcal{E})$ is a **level-weak equivalence** if $f_n : x_n \rightarrow y_n$ is in w for any integer n . We denote the class of level-weak equivalences by lw .

(2) Assume that $\mathbf{E} = (\mathcal{E}, w)$ is a strict relative exact category. We define the **bounded derived category of \mathbf{E}** by $\mathcal{D}_b(\mathbf{E}) = \mathcal{D}_b(\mathcal{E}, w) := \text{Coker}(\mathcal{D}_b(\mathcal{E}^w) \rightarrow \mathcal{D}_b(\mathcal{E}))$.

(3) In the situation (2), a morphism in $\mathbf{Ch}_b(\mathcal{E})$ is said to be a **quasi-weak equivalence** if its image in $\mathcal{D}_b(\mathbf{E})$ is an isomorphism. We denote the class of quasi-weak equivalences in $\mathbf{Ch}_b(\mathcal{E})$ by qw and put $\mathbf{Ch}_b(\mathbf{E}) := (\mathbf{Ch}_b(\mathcal{E}), qw)$. This association defines the 2-functor $\mathbf{Ch}_b : \mathbf{RelEx}_{\text{strict}}^+ \rightarrow \mathbf{RelEx}^+$.

(4) Let $f : \mathbf{E} \rightarrow \mathbf{F}$ be a morphism of strict relative exact categories. We say that f is a **derived equivalence** (resp. **weakly derived equivalence**, **derived fully faithful**) if induced functor on bounded derived categories is an equivalence of triangulated category (resp. equivalence of up to factor, fully faithful).

(5) Let \mathcal{R} be a subcategory of $\mathbf{RelEx}_{\text{strict}}$. We denote the class of derived equivalences in \mathcal{R} by $\text{deq}_{\mathcal{R}}$ or shortly deq . We call the (large) relative category $(\mathcal{R}, \text{deq}_{\mathcal{R}})$ the **homotopy theory of relative exact categories in \mathcal{R}** .

Remark 2.6. Recall the functor $P_{\mathcal{E}} : \mathbf{Ch}_b(\mathcal{E}) \rightarrow \mathcal{H}_b(\mathcal{E})$ is the canonical projection functor. We have the formula

$$\mathcal{D}_b(\mathcal{E}, w) = \mathcal{H}_b(\mathcal{E}) / (\mathcal{H}_b(\mathcal{E}^w) \vee_{\text{thi}} P_{\mathcal{E}}(\text{Acy}_b(\mathcal{E})))$$

by 1.17. We put

$$\text{Acy}_b^{qw}(\mathcal{E}) := P_{\mathcal{E}}^{-1}(\mathcal{H}_b(\mathcal{E}^w) \vee_{\text{thi}} P_{\mathcal{E}}(\text{Acy}_b(\mathcal{E}))),$$

$$\text{Acy}_b^{lw}(\mathcal{E}) := \mathbf{Ch}_b(\mathcal{E}^w).$$

Recall the definition of (weakly) exact sequences of triangulated categories from Conventions (8) (vi).

Definition 2.7 (Exact sequences of relative categories). (1) A sequence $\mathbf{E} \xrightarrow{u} \mathbf{F} \xrightarrow{v} \mathbf{G}$ of strict relative exact categories is **exact** (resp. **weakly exact**) if induced sequence of triangulated categories $\mathcal{D}_b(\mathbf{E}) \xrightarrow{\mathcal{D}_b(u)} \mathcal{D}_b(\mathbf{F}) \xrightarrow{\mathcal{D}_b(v)} \mathcal{D}_b(\mathbf{G})$ is exact (resp. weakly exact). We sometimes denote the sequence above by (u, v) . For a full subcategory \mathcal{R} of $\mathbf{RelEx}_{\text{strict}}^\#$, we let $E(\mathcal{R})$ (resp. $E_{\text{weak}}(\mathcal{R})$) denote the category of exact sequences (resp. weakly exact sequences) in \mathcal{R} as the full subcategory of $\mathcal{R}^{[2]}$. We define three functors $s^{\mathcal{R}}, m^{\mathcal{R}}$ and $q^{\mathcal{R}}$ from $E_{\#}(\mathcal{R})$ to \mathcal{R} which sends weakly exact sequence $\mathbf{E} \rightarrow \mathbf{F} \rightarrow \mathbf{G}$ to \mathbf{E} , \mathbf{F} and \mathbf{G} respectively. (2) Let \mathcal{R} and \mathcal{R}' be full subcategories of $\mathbf{RelEx}_{\text{strict}}$. A functor $F : \mathcal{R} \rightarrow \mathcal{R}'$ is **exact** (resp. **weakly exact**) if it sends an exact (resp. a weakly exact) sequence in \mathcal{R} to an exact (resp. a weakly exact) sequence in \mathcal{R}' .

Proposition 2.8 (Example of weakly exact sequences). Let \mathcal{E} be an exact category and v and w classes of morphisms in \mathcal{E} such that $v \subset w$ and both (\mathcal{E}, v) and (\mathcal{E}, w) are very strict relative exact categories. Then the inclusion functor $\mathcal{E}^w \hookrightarrow \mathcal{E}$ and the identity functor of \mathcal{E} induce a weakly exact sequence

$$(\mathcal{E}^w, v) \rightarrow (\mathcal{E}, v) \rightarrow (\mathcal{E}, w).$$

Proof. We apply 1.15 (1) to the fully faithful functors $\mathcal{D}_b(\mathcal{E}^v) \rightarrow \mathcal{D}_b(\mathcal{E}^w) \rightarrow \mathcal{D}_b(\mathcal{E})$. We learn that the induced functor $\mathcal{D}_b(\mathcal{E}^w, v) = \frac{\mathcal{D}_b(\mathcal{E})}{\mathcal{D}_b(\mathcal{E}^v)} \rightarrow \mathcal{D}_b(\mathcal{E}, v) = \frac{\mathcal{D}_b(\mathcal{E})}{\mathcal{D}_b(\mathcal{E}^v)}$ is fully faithful. On the other hand, we have an equivalence of triangulated categorie $\mathcal{D}_b(\mathcal{E}, w) = \frac{\mathcal{D}_b(\mathcal{E})}{\mathcal{D}_b(\mathcal{E}^w)} \xrightarrow{\sim} \frac{\frac{\mathcal{D}_b(\mathcal{E})}{\mathcal{D}_b(\mathcal{E}^v)}}{\frac{\mathcal{D}_b(\mathcal{E}^w)}{\mathcal{D}_b(\mathcal{E}^v)}}$ which makes the diagram below commutative

$$\begin{array}{ccc} & \mathcal{D}_b(\mathcal{E})/\mathcal{D}_b(\mathcal{E}^v) & \\ \swarrow & & \searrow \\ \mathcal{D}_b(\mathcal{E})/\mathcal{D}_b(\mathcal{E}^w) & \xrightarrow{\sim} & \frac{\mathcal{D}_b(\mathcal{E})/\mathcal{D}_b(\mathcal{E}^v)}{\mathcal{D}_b(\mathcal{E}^w)/\mathcal{D}_b(\mathcal{E}^v)} \end{array}$$

where all functors above are induced from the identity functor of \mathcal{E} . Hence we obtain the result. \square

Lemma-Definition 2.9 (Consistent axiom). For any strict relative exact category $\mathbf{E} = (\mathcal{E}, w)$, the following two conditions are equivalent.

- (1) $lw \subset qw$.
- (2) The canonical functor $j_{\mathcal{E}} : \mathcal{E} \rightarrow \mathbf{Ch}_b(\mathcal{E})$ is a relative exact functor $\mathbf{E} \rightarrow \mathbf{Ch}_b(\mathbf{E})$.

In this case, we say that w (or $\mathbf{E} = (\mathcal{E}, w)$) satisfies the **consistent axiom** or w (or \mathbf{E}) is consistent. We write $\mathbf{RelEx}_{\text{consist}}^\#$ (resp. $\mathbf{WalEx}_{\text{consist}}^\#$) for the full 2-subcategory of consistent relative exact categories (resp. consistent Waldhausen exact categories) in $\mathbf{RelEx}^\#$ (resp. $\mathbf{WalEx}^\#$) for $\# \in \{+, \text{nothing}\}$.

Proof. We can easily check that condition (1) implies condition (2). Assuming condition (2), we prove condition (1). Let $f : x \rightarrow y$ be a morphism in $\mathbf{Ch}_b(\mathcal{E})$. First let us notice that obviously the class qw is closed under the degree shift. Namely if f is in qw , then $f[n]$ is also

in qw for any integer n . In 3.20, we will prove that qw satisfies the extensional axiom. Suppose that f is in lw , then by the following short exact sequences

$$\begin{array}{ccccc} \sigma_{\leq n} x & \longrightarrow & x & \longrightarrow & \sigma_{\geq n+1} x \\ \sigma_{\leq n} f \downarrow & & f \downarrow & & \downarrow \sigma_{\geq n+1} f \\ \sigma_{\leq n} y & \longrightarrow & y & \longrightarrow & \sigma_{\geq n+1} y, \end{array}$$

induction on the length of chain complexes and by the extensional axiom for qw , we get the desired result. \square

Example 2.10. (1) A Quillen exact category \mathcal{E} with the class of all isomorphisms $(\mathcal{E}, i_{\mathcal{E}})$ is a consistent Waldhausen exact category.

(2) We will prove in 3.20 and 4.13 that for any strict relative exact category \mathbf{E} , $\mathbf{Ch}_b(\mathbf{E})$ is a complicial Waldhausen category. In particular, the category of bounded chain complexes on a Quillen exact category \mathcal{E} with the class of all quasi-isomorphisms $(\mathbf{Ch}_b(\mathcal{E}), \text{qis})$ is a consistent Waldhausen exact category.

(3) Let A be a Cohen-Macaulay ring and p a non-negative integer less than $\dim A$. Let us denote the category of finitely generated A -modules M whose codimension is greater than p by \mathcal{M}_A^p and the full subcategory of \mathcal{M}_A^p consisting of those A -modules such that its projective dimension is less than p by $\mathcal{M}_A^p(p)$. Then one can easily prove that $\mathcal{M}_A^p(p)$ is closed under extensions in \mathcal{M}_A^p and therefore it naturally becomes a Quillen exact category. A morphism $f : x \rightarrow y$ in $\mathcal{M}_A^p(p)$ is a **generic isomorphism** if the codimensions of $\text{Ker } f$ and $\text{Coker } f$ are greater than $p + 1$. We denote the class of generic isomorphisms in $\mathcal{M}_A^p(p)$ by w . Then one can easily prove that w satisfies the extensional axiom and $(\mathcal{M}_A^p(p))^w = \{0\}$. Therefore $qw = \text{qis}$ and obviously w does not satisfy the consistent axiom.

Recall the definition of the category of admissible exact sequences in a category with cofibrations from Conventions (7) (xiv).

Lemma-Definition 2.11. (1) Let \mathcal{G} and \mathcal{H} be strict exact subcategories of \mathcal{E} and we put $\mathbf{G} := (\mathcal{G}, w \cap \mathcal{G})$ and $\mathbf{H} := (\mathcal{H}, w \cap \mathcal{H})$. Let $E(\mathbf{G}, \mathbf{E}, \mathbf{H})$ denote the relative exact category $(E(\mathcal{G}, \mathcal{E}, \mathcal{H}), E(w))$. If \mathbf{E} , \mathbf{G} and \mathbf{H} are strict, then $E(\mathbf{G}, \mathbf{E}, \mathbf{H})$ is also.

(2) We write $\mathcal{HOM}(\mathbf{E}, \mathbf{F})$ for the relative category $(\mathcal{HOM}_{\mathbf{RelEx}^+}(\mathbf{E}, \mathbf{F}), \text{Mor } \mathcal{HOM}_{\mathbf{RelEx}^+}(\mathbf{E}, \mathbf{F}))$. In other words, $\mathcal{HOM}(\mathbf{E}, \mathbf{F})$ is a relative category whose underlying category is the category of relative exact functors from \mathbf{E} to \mathbf{F} and whose weak equivalences are relative natural equivalences. If \mathbf{F} is a Waldhausen exact category, then $\mathcal{HOM}(\mathbf{E}, \mathbf{F})$ is a relative exact category. Here a sequence $f \xrightarrow{a} g \xrightarrow{b} h$ of relative exact functors from \mathbf{E} to \mathbf{F} is an admissible exact sequence if for any object x in \mathcal{E} , a sequence $f(x) \xrightarrow{a(x)} g(x) \xrightarrow{b(x)} h(x)$ is an admissible exact sequence in \mathcal{F} .

(3) If \mathbf{F} is consistent, then the functor $\mathbf{Ch}_b : \mathcal{HOM}_{\mathbf{RelEx}^+}(\mathbf{E}, \mathbf{F}) \rightarrow \mathcal{HOM}_{\mathbf{RelEx}^+}(\mathbf{Ch}_b(\mathbf{E}), \mathbf{Ch}_b(\mathbf{F}))$ preserves relative natural weak equivalences. In particular, if \mathbf{F} is a consistent Waldhausen exact category, then the functor $\mathbf{Ch}_b : \mathcal{HOM}(\mathbf{E}, \mathbf{F}) \rightarrow \mathcal{HOM}(\mathbf{Ch}_b(\mathbf{E}), \mathbf{Ch}_b(\mathbf{F}))$ is a relative exact functor.

Proof. (1) We have an equality $E(\mathcal{G}, \mathcal{E}, \mathcal{H})^{E(w)} = E(\mathcal{G}^w, \mathcal{E}^w, \mathcal{H}^w)$. Hence if \mathbf{E} , \mathbf{G} and \mathbf{H} are strict, then $E(\mathbf{G}, \mathbf{E}, \mathbf{H})$ is strict.

(2) If $w = i_{\mathcal{E}}$ and $v = i_{\mathcal{F}}$, then $\text{Mor } \mathcal{HOM}_{\mathbf{RelEx}^+}(\mathbf{E}, \mathbf{F})$ is $i_{\mathcal{HOM}_{\mathbf{RelEx}^+}(\mathbf{E}, \mathbf{F})}$ and in this case, the assertion was essentially proven in [Moc10, A.11, A.18]. For general case, only non-trivial point is that for any diagrams $g \leftarrow h \rightarrow f$ and $g' \rightarrow h' \leftarrow f'$ of relative exact functors from \mathbf{E}

to \mathbf{F} , the functors $g \sqcup_h f, g' \times_{h'} f' : \mathcal{E} \rightarrow \mathcal{F}$ preserve weak equivalences. Let $a : x \rightarrow y$ be a morphism in w , then $g(a) \sqcup_{h(a)} f(a)$ and $g'(a) \times_{h'(a)} f'(a)$ are in v by the gluing and cogluing axioms. Hence we obtain the result.

(3) For each natural transformation θ between relative exact functors $f, g : \mathbf{E} = (\mathcal{E}, w) \rightarrow \mathbf{F} = (\mathcal{F}, v)$, if θ is a relative natural equivalence, $\mathbf{Ch}_b(\theta) : \mathbf{Ch}_b(f) \rightarrow \mathbf{Ch}_b(g)$ is contained in lv . Therefore if v satisfies the consistent axiom, then $\mathbf{Ch}_b(\theta)$ is a relative natural transformation between the morphisms $\mathbf{Ch}_b(f), \mathbf{Ch}_b(g) : \mathbf{Ch}_b(\mathbf{E}) \rightarrow \mathbf{Ch}_b(\mathbf{F})$. \square

3 Widely exact functors

The purpose of this section is to prove that for any strict relative exact category \mathbf{E} , $\mathbf{Ch}_b(\mathbf{E})$ is a Waldhausen exact category which satisfies the extensional, the saturated, the factorization and the retraction axioms in 3.20. The key tools is the concept about *widely exact functors* which is roughly saying exact functors from suitable exact categories to triangulated categories. Here “suitable” exact category means exact categories like the categories of complexes, namely *bicomplcial categories* in 3.1. We review the notion of *null classes* of a bicomplcial category which is bicomplcial variant of triangulated subcategories in 3.5 and study functorial behaviour of lattices of null classes and lattices of triangulated subcategories by widely exact functors in 3.14 and 3.15. At the end of this section, we will define the non-connective K -theory for consistent relative exact categories in 3.22 and prove that it is a categorical homotopy invariant functor in 3.25. We start by recalling the foundation of bicomplcial categories theory from [Moc10].

Definition 3.1 (Bicomplcial category). (1) A **bicomplcial category** is a system $(\mathcal{E}, C, r, \iota, \sigma)$ consisting of a Quillen exact category \mathcal{E} , an exact endofunctor $C : \mathcal{E} \rightarrow \mathcal{E}$, natural transformations $\iota : \text{id}_{\mathcal{E}} \rightarrow C, r : CC \rightarrow C$ and a natural equivalence $\sigma : CC \xrightarrow{\sim} CC$ which satisfies the following axioms:

- (i) We have the equalities $rC(\iota) = r\iota_C = \text{id}_C, \sigma C(\iota) = \iota_C$ and $\sigma\sigma = \text{id}_{CC}$.
- (ii) For any object x in \mathcal{E} , the morphism $\iota_x : x \rightarrow C(x)$ is an admissible monomorphism.

Then we put $T := C/\text{id}_{\mathcal{E}}$ and call it the **suspension functor**.

- (iii) T is essentially surjective and fully faithful.

We often omit C, ι, r and σ in the notation.

(2) A **complicial functor** between bicomplcial categories $\mathcal{E} \rightarrow \mathcal{E}'$ is a pair of an exact functor $f : \mathcal{E} \rightarrow \mathcal{E}'$ and a natural equivalence $c : C_{\mathcal{E}'} f \xrightarrow{\sim} f C_{\mathcal{E}}$ which satisfies the equality $cl_{\mathcal{E}'}^{\mathcal{E}'} = f(\iota_{\mathcal{E}})$. We often omit c in the notation.

(3) For complicial functors $\mathcal{E} \xrightarrow{(f,c)} \mathcal{E}' \xrightarrow{(g,d)} \mathcal{E}''$, their composition is defined by $(g,d)(f,c) := (gf, d \odot c)$ where we put $d \odot c := g(c)d_f$.

(4) A **complicial natural transformation** between complicial functors $(f, c), (g, d) : \mathcal{E} \rightarrow \mathcal{E}'$ from (f, c) to (g, d) is a natural transformation $\phi : f \rightarrow g$ which subjects to the condition that $dC\phi = \phi_C c$.

(5) We denote the 2-category of essentially small bicomplcial categories by **BiComp**.

Example 3.2 (Bicomplcial categories). (1) Let \mathcal{E} be a Quillen exact category. Now we give the bicomplcial structure on $\mathbf{Ch}_{\#}(\mathcal{E})$ ($\# = b, \pm, \emptyset$) as follows. The functor $C : \mathbf{Ch}_b(\mathcal{E}) \rightarrow \mathbf{Ch}_b(\mathcal{E})$ is given by $x \mapsto Cx := \text{Cone id}_x$ and for any complex x , we define morphisms $(\iota_x) : x \rightarrow C(x), (r_x) : CC(x) \rightarrow C(x)$ and $(\sigma_x) : CC(x) \rightarrow CC(x)$ by

$$(\iota_x)_n = \begin{pmatrix} 0 \\ \text{id}_{x_n} \end{pmatrix}, (r_x)_n = \begin{pmatrix} 0 & \text{id}_{x_{n-1}} & \text{id}_{x_{n-1}} & 0 \\ 0 & 0 & 0 & \text{id}_{x_n} \end{pmatrix} \text{ and } (\sigma_x)_n = \begin{pmatrix} -\text{id}_{x_{n-2}} & 0 & 0 & 0 \\ 0 & 0 & \text{id}_{x_{n-1}} & 0 \\ 0 & \text{id}_{x_{n-1}} & 0 & 0 \\ 0 & 0 & 0 & \text{id}_{x_n} \end{pmatrix}.$$

Then the system $(\mathbf{Ch}_b(\mathcal{E}), C, \iota, r, \sigma)$ forms a bicomplial category.

(2) More generally, a complicial exact category in [Sch11, 3.2.2] can be regarded as a bicomplial category as follows. Let $\mathbf{Ch}_b(\mathbb{Z})$ be the category of bounded chain complexes of finitely generated free abelian groups. (But in Ibid, we use the cohomological notation for complexes). Here the bilinear operation \otimes on $\mathbf{Ch}_b(\mathbb{Z})$ is given by the usual tensor products of complexes of abelian groups. (For more detail, see Ibid). We can easily learn that a complex in $\mathbf{Ch}_b(\mathbb{Z})$ is isomorphic to finite direct sum of degree shift of the following two typical complexes: The unit complex $\mathbb{1}$ is \mathbb{Z} in degree 0 and 0 elsewhere. For any positive integer m , the complex $C_m := [\mathbb{Z} \xrightarrow{m} \mathbb{Z}]$ is \mathbb{Z} in degrees 0 and 1 and the only non-trivial boundary map d_1 is given by multiplication of m . The complex C_1 has the natural commutative monoid object structure $I : \mathbb{1} \rightarrow C_1$, $R : C_1 \otimes C_1 \rightarrow C_1$ and $s : C_1 \otimes C_1 \rightarrow C_1 \otimes C_1$ which is implicitly explained in (1). In particular, if an exact category \mathcal{C} is complicial in the sense of [Sch11], that is, there is an action on \mathcal{C} by a biexact functor $- \otimes - : \mathbf{Ch}_b(\mathbb{Z}) \times \mathcal{C} \rightarrow \mathcal{C}$, it defines the bicomplial structure on \mathcal{C} by $C := C_1 \otimes -$, $\iota := I \otimes \text{id}_-$, $r := R \otimes \text{id}_-$ and $\sigma := s \otimes \text{id}_-$. As we showed in [Moc10], the almost arguments about complicial exact categories and complicial Waldhausen categories in [Sch11] is essentially only using the commutative monoid object structure of C_1 and still works fine in the context of bicomplial categories and bicomplial pairs which will be defined in 3.7. Therefore, in this paper, to make the definitions simplify, we review and utilize the theory of complicial exact categories (with weak equivalences) in [Sch11] as the theory of bicomplial categories and bicomplial pairs with slight modifications.

(3) Recall the notations of morphisms categories from Conventions (5) (ii). If \mathcal{E} is a bicomplial category, then $\text{Ar } \mathcal{E}$ naturally becomes a bicomplial category and moreover (dom, id) , $(\text{ran}, \text{id}) : \text{Ar } \mathcal{E} \rightarrow \mathcal{E}$ are complicial functors and ϵ is a complicial natural transformation.

Notations 3.3 (C -homotopy equivalences). Let $(\mathcal{E}, C, r, \iota, \sigma)$ be a bicomplial category. (1) Morphisms $f, g : x \rightarrow y$ in \mathcal{E} are **(C -)homotopic** if there exists a morphism $H : Cx \rightarrow y$ such that $f - g = H\iota_x$.

(2) A morphism $f : x \rightarrow y$ in \mathcal{E} is a **(C -)homotopy equivalence** if there exists a morphism $g : y \rightarrow x$ such that gf and fg are homotopic to id_x and id_y respectively. Then we say that x and y are **C -homotopy equivalent**.

(3) An object x is **(C -)contractible** if x is C -homotopy equivalent to the zero object. Namely id_x is C -homotopic to the zero morphism.

(4) We can easily prove that C -homotopic is an equivalence relation on Hom -sets of \mathcal{E} which is compatible with the composition. (See [Moc10, 2.13]). Therefore we can define the **homotopy category** $\pi_0(\mathcal{E})$ of \mathcal{E} as follows. The object class of $\pi_0(\mathcal{E})$ is same as that of \mathcal{E} , the class of morphisms in $\pi_0(\mathcal{E})$ is the homotopic class of morphisms in \mathcal{E} and the composition of morphisms is induced from \mathcal{E} . We can easily prove that $\pi_0(\mathcal{E})$ is an additive category.

Notations 3.4 (Mapping cone functor, Mapping cylinder functor). Let (\mathcal{E}, C) be a bicomplial category.

(1) We define the functors

$$\text{Cone}, \text{Cyl} : \text{Ar } \mathcal{E} \rightarrow \mathcal{E}$$

called the **mapping cone functor** and the **mapping cylinder functor** respectively as follows:

$$\text{Cyl} := \text{ran} \oplus C \text{ dom}$$

$$\text{Cone} := \text{ran} \sqcup_{\text{dom}} C \text{ dom}$$

where Cone is defined the following push out diagram:

$$\begin{array}{ccccc} \text{dom} & \xrightarrow{\iota_{\text{dom}}} & C \text{ dom} & \xrightarrow{\pi_{\text{dom}}} & T \text{ dom} \\ \epsilon \downarrow & \star & \downarrow \mu & & \downarrow \text{id}_{T \text{ dom}} \\ \text{ran} & \xrightarrow{\kappa} & \text{Cone} & \xrightarrow{\psi} & T \text{ dom} \end{array}$$

where ψ is induced from the universal property of Cone . We have the conflation

$$\text{dom} \xrightarrow{j_1} \text{Cyl} \xrightarrow{\eta} \text{Cone}$$

where $j_1 := \begin{pmatrix} \epsilon \\ -\iota_{\text{dom}} \end{pmatrix}$ and $\eta = (\kappa \quad \mu)$.

(2) Moreover we define the natural transformations $j_2 : \text{ran} \rightarrow \text{Cyl}$, $j_3 : C \text{ dom} \rightarrow \text{Cyl}$ and $\beta : \text{Cyl} \rightarrow \text{ran}$ by $j_2 := \begin{pmatrix} \text{id}_{\text{ran}} \\ 0 \end{pmatrix}$, $j_3 := \begin{pmatrix} 0 \\ \text{id}_{C \text{ dom}} \end{pmatrix}$ and $\beta := (\text{id}_{\text{ran}} \quad 0)$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} \text{dom} & \xrightarrow{j_1} & \text{Cyl} & \xleftarrow{j_2} & \text{ran} \\ & \searrow \epsilon & \downarrow \beta & \swarrow \text{id}_{\text{ran}} & \\ & & \text{ran} & & \end{array}$$

(3) By declaring that T -sequence is T -exact if it is isomorphic to the following type T -exact triangle

$$x \xrightarrow{f} y \xrightarrow{\kappa_f} \text{Cone } f \xrightarrow{\psi_f} Tx,$$

we can make $(\pi_0(\mathcal{E}), T)$ into a triangulated category. (cf. [Moc10, 3.29]).

3.5 (Null classes and complicial weak equivalences). (cf. [Moc10, §5.1]). Let (\mathcal{E}, C) be a bicomplial category.

(1) We say that a full subcategory \mathcal{N} of \mathcal{E} is a **null class** if it contains all C -contractible objects in \mathcal{E} and for any admissible short exact sequences $x \rightarrowtail y \twoheadrightarrow z$ in \mathcal{E} , if two of x , y and z are in \mathcal{N} , then the other is also in \mathcal{N} . We call the last condition the **two out of three for admissible short exact sequences axiom**.

(2) We say that a null class of \mathcal{N} is **thick** if \mathcal{N} is closed under retractions. In this case, $\pi_0(\mathcal{N})$ is a thick triangulated subcategory of $\pi_0(\mathcal{E})$.

(3) If \mathcal{E} is essentially small, we denote the set of null classes (resp. thick null classes) in \mathcal{E} by $\text{NC}_\emptyset(\mathcal{E})$ or $\text{NC}(\mathcal{E})$ (resp. $\text{NC}_{\text{thi}}(\mathcal{E})$).

(4) A class of morphisms w in \mathcal{E} is a **class of complicial weak equivalences** if it satisfies the saturated, the extensional axioms and if it contains all C -homotopy equivalences. We call a morphism in w a **complicial weak equivalence**.

(5) A class of complicial weak equivalences is **thick** if it satisfies the retraction axiom.

(6) If \mathcal{E} is essentially small, we denote the set of classes of complicial weak equivalences (resp. thick classes of complicial weak equivalences) in \mathcal{E} by $\text{CW}_\emptyset(\mathcal{E})$ or $\text{CW}(\mathcal{E})$ (resp. $\text{CW}_{\text{thi}}(\mathcal{E})$).

(7) We can easily check that $\text{CW}_\#(\mathcal{E})$ and $\text{NC}_\#(\mathcal{E})$ are complete lattices by 1.1 where $\# = \emptyset$ or thi . Therefore they define the functors.

$$\text{NC}_\#, \text{CW}_\# : \text{BiComp}^{\text{op}} \rightarrow \text{CLat}.$$

where for any complicial functor $f : \mathcal{E} \rightarrow \mathcal{E}'$, $\text{NC}_\#(f)$ and $\text{CW}_\#(f)$ are defined by pull-back by f .

(8) For any full subcategory \mathcal{N} of \mathcal{E} , we write $\mathcal{N}_{\text{nul}, \mathcal{E}}$ or simply \mathcal{N}_{nul} (resp. $\mathcal{N}_{\text{thi}, \mathcal{E}}$ or simply \mathcal{N}_{thi}) for the smallest null class (resp. thick null class) containing \mathcal{N} and call it the **null** (resp. **thick**) **closure of \mathcal{N}** . For any class of morphisms v in \mathcal{E} , we write $v_{\text{comp}, \mathcal{E}}$ or simply v_{comp} (resp. $v_{\text{thi}, \mathcal{E}}$ or simply v_{thi}) for the smallest class of complicial weak equivalences (resp. thick complicial weak equivalences) containing v and call it the **complicial** (resp. **thick complicial**) **closure of v** .

(9) For any full subcategory \mathcal{N} of \mathcal{E} , we put

$$w_{\mathcal{N}} := \{f \in \text{Mor } \mathcal{E}; \text{Cone } f \in \mathcal{N}\}.$$

Then the association $\mathcal{N} \mapsto w_{\mathcal{N}}$ gives the natural equivalence $\mathbf{NC}_{\#} \xrightarrow{\sim} \mathbf{CW}_{\#}$ ($\# = \emptyset$ or thi). Here the inverse is given by $w \mapsto \mathcal{E}^w$.

(10) For any class of complicial weak equivalences w , \mathcal{E}^w is w -closed in the following sense. For any objects x and y in \mathcal{E} , if x is in \mathcal{E}^w and if there exists a morphism $x \rightarrow y$ or $y \rightarrow x$ in w , then y is also in \mathcal{E}^w . For in this situation, by the saturated axiom, the assertion that $0 \rightarrow x$ is in w implies the assertion that $0 \rightarrow y$ is in w .

3.6 (Frobenius exact structure). (cf. [Sch11, 6.5, B.16], [Moc10, 2.26, 2.27, 2.30, 2.40]). Let us recall that a Quillen exact category \mathcal{F} is **Frobenius** if the class of projective objects in \mathcal{F} is equal to that of injective objects in \mathcal{F} and if \mathcal{F} has enough proj-inj objects. For any bicomplcial category (\mathcal{E}, C) , it naturally has the Frobenius exact category structure as follows. An admissible monomorphism (resp. admissible epimorphism) $x \xrightarrow{a} y$ is **Frobenius** if for any object u and a morphism $x \xrightarrow{f} Cu$ (resp. $Cu \xrightarrow{f} y$), there exists a morphism $y \xrightarrow{g} Cu$ (resp. $Cu \xrightarrow{g} x$) such that $f = ga$ (resp. $f = ag$). In an admissible short exact sequence $x \xrightarrow{i} y \xrightarrow{p} z$ in \mathcal{E} , i is Frobenius if and only if p is. In this case we call the sequence $x \xrightarrow{i} y \xrightarrow{p} z$ a **Frobenius admissible short exact sequence**. The Quillen exact category \mathcal{E} with Frobenius admissible short exact sequences forms a Frobenius exact category and we denote it by $\mathcal{E}_{\text{frob}}$. An object in $\mathcal{E}_{\text{frob}}$ is a proj-inj object if and only if it is a C -contractible object. Moreover $(\mathcal{E}_{\text{frob}}, C)$ again becomes a bicomplcial category. For example, for any Quillen exact category \mathcal{F} , a Frobenius admissible short exact sequence in the standard bicomplcial category $\mathbf{Ch}_b(\mathcal{F})$ is just a degree-wised split exact sequence.

Definition 3.7 (Bicomplcial pair). (1) A **bicomplcial pair** is a pair $\mathbf{C} = (\mathcal{C}, w)$ of bicomplcial category \mathcal{C} and a class of complicial weak equivalences w in \mathcal{C} .

(2) A **relative complicial functor** between bicomplcial categories $\mathbf{C} \rightarrow \mathbf{C}'$ is a complicial functor such that the underlying functor preserves weak equivalences.

(3) A **relative complicial natural transformation** between relative complicial functors is just a complicial natural transformation.

(4) A **relative complicial natural weak equivalence** is a relative complicial natural transformation such that the underlying natural transformation is a relative natural equivalence.

(5) We denote the 2-category of essentially small bicomplcial pairs, relative complicial functors and relative complicial natural transformations (resp. relative complicial natural weak equivalences) by $\mathbf{BiCompPair}^+$ (resp. $\mathbf{BiCompPair}$).

(6) A bicomplcial pair $\mathbf{E} = (\mathcal{E}, w)$ is **thick** if w is thick. We write $\mathbf{BiCompPair}_{\text{thi}}^{\#}$ for the full 2-subcategory of thick bicomplcial pairs in $\mathbf{BiCompPair}^{\#}$ for $\# \in \{+, \text{nothing}\}$.

(7) For any bicomplcial pair $\mathbf{C} = (\mathcal{C}, w)$, $\pi_0(\mathcal{C}^w)$ is a triangulated subcategory of $\pi_0(\mathcal{C})$ by 3.5 (2) and (9) and we put $\mathcal{T}(\mathbf{C}) = \mathcal{T}(\mathcal{C}, w) := \pi_0(\mathcal{C})/\pi_0(\mathcal{C}^w)$. This association defines the 2-functor $\mathcal{T} : \mathbf{BiCompPair}^+ \rightarrow \mathbf{TriCat}$ which sends relative complicial natural weak equivalences to triangulated natural equivalences.

Remark 3.8. (1) (cf. [Moc10, 5.18]). A bicomplcial pair is a saturated extensional Waldhausen exact category which satisfies the factorization axiom.

(2) (cf. [Moc10, 5.3]). For a thick bicomplcial pair (\mathcal{C}, w) , $\pi_0(\mathcal{C}^w)$ is a thick subcategory of $\pi_0(\mathcal{C})$.

Recall the definition of very strict relative exact categories from 2.2 (5).

Proposition 3.9. *A bicomplcial pair is very strict.*

To prove Proposition 3.9, we utilize the following lemma.

Lemma 3.10. (cf. [Sch11, 3.1.7 (b)]). *Let \mathcal{E} be an exact category and $\mathcal{F} \xrightarrow{i} \mathcal{E}$ a strict exact subcategory of \mathcal{E} . We assume that the following condition $(*)$ holds. Then i induces a fully faithful functor $\mathcal{D}_b(\mathcal{F}) \rightarrow \mathcal{D}_b(\mathcal{E})$.*

(*) For any admissible monomorphism $x \xrightarrow{a} y$ in \mathcal{E} with $x \in \text{Ob } \mathcal{F}$, there exists a morphism $y \xrightarrow{b} z$ with $z \in \text{Ob } \mathcal{F}$ such that the composition $ba : x \rightarrow z$ is an admissible monomorphism. \square

Proof of 3.9. Let $\mathbf{C} = (\mathcal{C}, w)$ be a bicomplial pair. We are going to check that $\mathcal{C}^w \hookrightarrow \mathcal{C}$ satisfies the condition (*) in 3.10. For any admissible monomorphism $x \rightarrow y$ with $x \in \text{Ob } \mathcal{C}^w$, the morphism $\iota_y : y \rightarrow C(y)$ is an admissible monomorphism with $C(y) \in \text{Ob } \mathcal{C}^w$. Hence we get the result by 3.10. \square

Recall the notations of $E(\mathcal{E})$ and $s^\mathcal{E}, q^\mathcal{E}$ from Conventions (7) (ii) and (xiv).

Definition 3.11 (Widely exact functors). Let \mathcal{E} be a bicomplial category and (\mathcal{T}, Σ) a triangulated category. A **widely exact functor** from \mathcal{E} to \mathcal{T} is a pair of (f, ∂) consisting of an additive functor $f : \mathcal{E} \rightarrow \mathcal{T}$ and a natural equivalence $\partial : f q^\mathcal{E} \rightarrow \Sigma f s^\mathcal{E}$ satisfies the following conditions:

- (1) For any admissible exact sequence $x \xrightarrow{i} y \xrightarrow{p} z$ in \mathcal{E} , $fx \xrightarrow{fi} fy \xrightarrow{fp} fz \xrightarrow{\partial(i,p)} \Sigma fx$ is a Σ -exact triangle.
- (2) For any x in \mathcal{E} , $\partial_x := \partial_{(\iota_x, \pi_x)} : fTx \rightarrow \Sigma fx$ is an isomorphism where $x \xrightarrow{\iota_x} Cx \xrightarrow{\pi_x} Tx$ is the canonical admissible exact sequence in \mathcal{E} .

We often omit ∂ in the notation. We write $\text{Ex}_{\text{wide}}(\mathcal{E}, \mathcal{T})$ for the set of widely exact functors from \mathcal{E} to \mathcal{T} .

Example 3.12 (Compositions). Let $(f, \partial) : \mathcal{E} \rightarrow \mathcal{T}$ be a widely exact functor from a bicomplial category \mathcal{E} to a triangulated category (\mathcal{T}, Σ) .

- (1) For a complicial functor $(f, c) : \mathcal{E}' \rightarrow \mathcal{E}$ from a bicomplial category \mathcal{E}' , we have the widely exact functor $gf : \mathcal{E}' \rightarrow \mathcal{T}$.
- (2) For a triangulated functor $(h, d) : \mathcal{T} \rightarrow \mathcal{T}'$ from \mathcal{T} to a triangulated category \mathcal{T}' , we have the widely exact functor $hf : \mathcal{E} \rightarrow \mathcal{T}'$.

Lemma 3.13 (Fundamental properties of widely exact functors). Let $(f, c) : \mathcal{E} \rightarrow \mathcal{T}$ be a widely exact functor from a bicomplial category (\mathcal{E}, C) to a triangulated category (\mathcal{T}, Σ) . Then

- (1) For any object x in \mathcal{E} , fCx is the zero object in \mathcal{T} .
- (2) For any morphisms $a, b : x \rightarrow y$ in \mathcal{E} , if a and b are homotopic, then we have the equality $fa = fb$.
- (3) If $x \xrightarrow{a} y$ is a homotopy equivalence, then fa is an isomorphism.
- (4) If an object x in \mathcal{E} is contractible, then fx is the zero object in \mathcal{T} .

Proof. (1) For any x in \mathcal{E} , by considering the diagram of Σ -exact triangles below

$$\begin{array}{ccccccc}
 fx & \xrightarrow{\iota_x} & fCx & \xrightarrow{\pi_x} & fTx & \xrightarrow{\partial_{(\iota_x, \pi_x)}} & \Sigma fx \\
 \parallel & & \downarrow & & \downarrow \wr & & \parallel \\
 fx & \longrightarrow & 0 & \longrightarrow & \Sigma fx & \xrightarrow{-\text{id}_{\Sigma fx}} & \Sigma fx,
 \end{array}$$

we learn that fCx is the zero object.

- (2) For any morphism $a, b : x \rightarrow y$, if there exists a homotopy $H : a \rightarrow b$, then we have the equalities $fa - fb = fHf\iota_x = 0$ by (1).

(3) For morphisms $a : x \rightarrow y$ and $b : y \rightarrow x$ such that ba and ab are homotopic to the identity morphisms, fb and fa are the inverse morphisms in each other by (2).

(4) Let x be a contractible object, then since the canonical morphism $x \rightarrow 0$ is a homotopy equivalence, $fx \rightarrow 0$ is an isomorphism in \mathcal{T} by (3). \square

Corollary 3.14. *Let $f : \mathcal{E} \rightarrow \mathcal{T}$ be a widely exact functor from a bicomplial category \mathcal{E} to a triangulated category \mathcal{T} .*

(1) *For a triangulated subcategory \mathcal{L} of \mathcal{T} , $f^{-1}\mathcal{L}$ is a null class in \mathcal{E} .*

(2) *In (1), moreover if \mathcal{L} is thick, then $f^{-1}\mathcal{L}$ is also thick.*

In particular, f induces the ordered map $\text{Tri}_{\#}(\mathcal{T}) \rightarrow \text{NC}_{\#}(\mathcal{E})$, $\mathcal{N} \mapsto f^{-1}\mathcal{N}$ where $\# = \emptyset$ or thi.

Proof. (1) For a contractible object x in \mathcal{C} , since $f(x)$ is the zero object, x is in $f^{-1}\mathcal{L}$. For an admissible short exact sequence $x \rightarrow y \rightarrow z$, since $fx \rightarrow fy \rightarrow fz \rightarrow \Sigma fx$ is Σ -exact, if two of fx , fy and fz are in \mathcal{L} , then third one is also in \mathcal{L} and therefore $f^{-1}\mathcal{L}$ is a null class. Assertion (2) follows from 1.2. \square

Proposition 3.15 (Widely exact functors). (1) *For a bicomplial category \mathcal{E} , the canonical functor*

$$\omega_{\mathcal{E}} : \mathcal{E}_{\text{frob}} \rightarrow \pi_0(\mathcal{E})$$

is widely exact.

(2) *In the situation above, let (\mathcal{T}, Σ) be a triangulated category. Then the association*

$$\omega_{\mathcal{E}}^* : \text{Hom}_{\text{TriCat}}(\pi_0(\mathcal{E}), \mathcal{T}) \ni g \mapsto g \circ \omega_{\mathcal{E}} \in \text{Ex}_{\text{wide}}(\mathcal{E}_{\text{frob}}, \mathcal{T})$$

gives a bijective correspondence. Moreover $\omega_{\mathcal{E}}$ induces the lattices isomorphism

$$\text{Tri}_{\#}(\pi_0(\mathcal{E})) \xrightarrow{\sim} \text{NC}_{\#}(\mathcal{E}_{\text{frob}}), \mathcal{N} \mapsto \omega_{\mathcal{E}}^{-1}\mathcal{N}.$$

(3) *For a bicomplial pair $\mathbf{E} = (\mathcal{E}, w)$, the canonical functor*

$$\tilde{\omega}_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{T}(\mathcal{E}, w)$$

is widely exact.

(4) *In the situation above, if \mathbf{E} is thick, then the canonical functor $\tilde{\omega}_{\mathcal{E}}$ induces the lattice isomorphism*

$$\{\mathcal{N} \in \text{NC}_{\#}(\mathcal{E}_{\text{frob}}); \mathcal{E}^w \subset \mathcal{N}\} \xrightarrow{\sim} \text{Tri}_{\#}(\mathcal{T}(\mathcal{E}, w)).$$

Proof. (1) is proven in [Moc10, 3.25].

(2) For any widely exact functor $(f, \partial) : \mathcal{E}_{\text{frob}} \rightarrow \mathcal{T}$, let us define the triangle functor $(\bar{f}, \bar{\partial}) : \pi_0(\mathcal{E}) \rightarrow \mathcal{T}$ by the formula $\bar{f}(\omega_{\mathcal{E}}a) := f(a)$ for any morphism $a : x \rightarrow y$ in \mathcal{E} . By virtue of 3.13 (2), this association is well-defined and the association $f \mapsto \bar{f}$ gives the inverse map of $\omega_{\mathcal{E}}^*$. For the second assertion, the association $\mathcal{N} \mapsto \pi_0(\mathcal{N})$ gives the inverse map of $\omega_{\mathcal{E}}^{-1}$.

(3) For an admissible short exact sequence $x \xrightarrow{i} y \xrightarrow{p} z$ in \mathcal{E} , let us consider the following

commutative diagram of admissible short exact sequences

$$\begin{array}{ccccc}
 x & \xrightarrow{j_1(i)} & \text{Cyl } i & \xrightarrow{\eta_i} & \text{Cone } i \\
 \parallel & & \downarrow \beta_i & & \downarrow \text{Cone}(0,p) \\
 x & \xrightarrow{i} & y & \xrightarrow{p} & z.
 \end{array}$$

Since β_i is a homotopy equivalence, it turns out that $\text{Cone}(0, p)$ is a weak equivalence. Therefore the sequence $x \xrightarrow{i} y \xrightarrow{p} z \xrightarrow{\psi_i \text{Cone}(0,p)^{-1}} Tx$ is a T -exact triangle in $\mathcal{T}(\mathbf{E})$.

(4) Since w is thick, $\pi_0(\mathcal{E}^w)$ is a thick subcategory of $\pi_0(\mathcal{E})$ by 3.8 (2). Therefore we have the lattice isomorphisms

$$\begin{aligned}
 \mathbf{NC}_{\#}(\mathcal{E}_{\text{frob}}) &\xrightarrow{\sim} \text{Tri}_{\#}(\pi_0(\mathcal{E})) \text{ and} \\
 V_{\#}(\pi_0(\mathcal{E}^w)) &\xrightarrow{\sim} \text{Tri}_{\#}(\mathcal{T}(\mathcal{E}, w))
 \end{aligned}$$

by (1) and 1.5. Hence we obtain the desired lattice isomorphism. \square

Corollary 3.16. *For any bicomplial pair (\mathcal{E}, w) , the identity functor $(\mathcal{E}, w) \rightarrow (\mathcal{E}, w_{\text{thi}, \mathcal{E}_{\text{frob}}})$ induces an equivalence of triangulated categories $\mathcal{T}(\mathcal{E}, w) \xrightarrow{\sim} \mathcal{T}(\mathcal{E}, w_{\text{thi}, \mathcal{E}_{\text{frob}}})$.*

Proof. By 3.15 (2), $\pi_0(\mathcal{E}^{w_{\text{thi}, \mathcal{E}_{\text{frob}}}}) = \pi_0(\mathcal{E}^w)_{\text{thi}}$. Therefore we have

$$\mathcal{T}(\mathcal{E}, w) = \pi_0(\mathcal{E})/\pi_0(\mathcal{E}^w) \xrightarrow{\sim} \pi_0(\mathcal{E})/\pi_0(\mathcal{E}^w)_{\text{thi}} = \mathcal{T}(\mathcal{E}, w_{\text{thi}, \mathcal{E}_{\text{frob}}}).$$

\square

Recall the definition of cokernel of triangulated functors from Conventions (8) (v).

Definition 3.17. (1) Let $f : \mathbf{E} = (\mathcal{E}, w) \rightarrow \mathbf{F} = (\mathcal{F}, v)$ be a relative exact functor between strict relative exact categories. Then we define the relative exact category $\text{Coker } f$ by $(\mathbf{Ch}_b(\mathcal{F}), w_f)$. Here the morphism $a : x \rightarrow y$ in $\mathbf{Ch}_b(\mathcal{F})$ is in w_f if and only if the image of a by composition of the canonical projection functors $\mathbf{Ch}_b(\mathcal{F}) \rightarrow \mathcal{D}_b(\mathcal{F}) \rightarrow \text{Coker } \mathcal{D}_b(f)$ is an isomorphism.

(2) If $f : \mathbf{E} \rightarrow \mathbf{F}$ is derived fully faithful, then we write \mathbf{F}/\mathbf{E} for $\text{Coker } f$.

(3) If \mathbf{F} is consistent, then the functor $j_{\mathcal{F}} : \mathcal{F} \rightarrow \mathbf{Ch}_b(\mathcal{F})$ induces a relative exact functor $\pi_f : \mathbf{F} \rightarrow \text{Coker } f$.

(4) Let

$$\begin{array}{ccc}
 \mathbf{E} & \xrightarrow{f} & \mathbf{F} \\
 a \downarrow & & \downarrow b \\
 \mathbf{E}' & \xrightarrow{f'} & \mathbf{F}'
 \end{array}$$

be a commutative diagram of strict relative exact categories. Then we define $\text{Coker}(a, b) : \text{Coker } f \rightarrow \text{Coker } f'$ to be a relative exact functor by $\text{Coker}(a, b) := \mathbf{Ch}_b(b)$.

Remark 3.18. (1) In 3.17 (1), we have the canonical equivalence of triangulated categories $\text{Coker } \mathcal{D}_b(f) \xrightarrow{\sim} \mathcal{T}(\mathbf{Ch}_b(\mathcal{F}), w_f)$.

(2) In 3.17 (4), if both \mathbf{F} and \mathbf{F}' are consistent, then the diagram below is commutative.

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{\pi_f} & \text{Coker } f \\ \downarrow b & & \downarrow \text{Coker}(a, b) \\ \mathbf{F}' & \xrightarrow{\pi_{f'}} & \text{Coker } f'. \end{array}$$

Example 3.19. For any strict relative exact category $\mathbf{E} = (\mathcal{E}, w)$, we have the equality

$$\mathbf{Ch}_b(\mathbf{E}) = \text{Coker}((\mathcal{E}^w, i_{\mathcal{E}^w}) \rightarrow (\mathcal{E}, i_{\mathcal{E}})).$$

Recall the definition about exact sequences of triangulated categories from Conventions (8) (vi).

Proposition 3.20. *Let $f : \mathbf{E} = (\mathcal{E}, w) \rightarrow \mathbf{F}(\mathcal{F}, v)$ be a relative exact functor between strict relative exact categories. Then*

(1) *$\text{Coker } f$ is a thick bicomplial pair. In particular, $\mathbf{Ch}_b(\mathbf{E})$ is a thick bicomplial pair. In particular, $\mathbf{Ch}_b(\mathbf{E})$ is a Waldhausen exact category which satisfies the extensional, the saturated, the factorization and the retraction axioms.*

(2) *The inclusion functor $\mathbf{Ch}_b(\mathcal{E})^{qw} \hookrightarrow \mathbf{Ch}_b(\mathcal{E})$ and the identity functor of $\mathbf{Ch}_b(\mathcal{E})$ induce an exact sequence of triangulated categories*

$$\mathcal{T}(\mathbf{Ch}_b(\mathcal{E})^{qw}, \text{qis}) \rightarrow \mathcal{T}(\mathbf{Ch}_b(\mathcal{E}), \text{qis}) \rightarrow \mathcal{T}(\mathbf{Ch}_b(\mathcal{E}), qw).$$

Proof. (1) w_f is corresponding to the null class $\text{Acy}_b^{w_f} \mathcal{F}$ which is the pull back of $\{0\}$ by the composition of the widely exact functor and the triangulated functors

$$\mathbf{Ch}_b(\mathcal{F}) \rightarrow \mathcal{D}_b(\mathcal{F}) \rightarrow \mathcal{D}_b(\mathbf{F}) \rightarrow \text{Coker } \mathcal{D}_b(f).$$

Therefore by 3.12 and 3.14, it is a class of thick complicial weak equivalences. The last assertion follows from 3.8.

(2) We only need to check that $\mathcal{T}(\mathbf{Ch}_b(\mathcal{E})^{qw}, \text{qis})$ is a thick subcategory of $\mathcal{T}(\mathbf{Ch}_b(\mathcal{E}), \text{qis})$. We apply 3.15 (4) to a thick bicomplial pair $(\mathbf{Ch}_b(\mathcal{E}), \text{qis})$. A thick null class $(\text{Acy}_b^b(\mathcal{E}) \subset) \mathbf{Ch}_b(\mathcal{E})^{qw}$ of $\mathbf{Ch}_b(\mathcal{E})_{\text{frob}}$ corresponds to a thick subcategory $\mathcal{T}(\mathbf{Ch}_b(\mathcal{E})^{qw}, \text{qis})$ of $\mathcal{T}(\mathbf{Ch}_b(\mathcal{E}), \text{qis})$. \square

Corollary 3.21. (1) *The 2-functor \mathbf{Ch}_b induces the 2-functor $\mathbf{RelEx}_{\text{strict}}^+ \rightarrow \mathbf{BiCompPair}_{\text{thi}}$.*
(2) *The functor $\mathcal{D}_b : \mathbf{RelEx}_{\text{strict}}^+ \rightarrow \mathbf{TriCat}$ is a 2-functor and the restriction $\mathbf{RelEx}_{\text{consist}}^+ \rightarrow \mathbf{TriCat}$ sends relative natural weak equivalences to triangulated natural weak equivalences. In particular, for a morphism $f : \mathbf{E} \rightarrow \mathbf{F}$ in $\mathbf{RelEx}_{\text{strict}}$, if F is consistent and if f is an exact homotopy equivalence, then it is a derived equivalence.*

Proof. Assertion (1) follows from 3.20 (1). Since \mathcal{D}_b is canonically isomorphic to $\mathcal{T} \mathbf{Ch}_b$, assertion (2) follows from 2.11 (3). \square

Now we give a definition of non-connective K -theory of consistent relative exact categories.

Definition 3.22 (Non-connective K -theory). For any consistent relative exact category $\mathbf{E} := (\mathcal{E}, w)$, we define the non-connective K -theory $\mathbb{K}(\mathbf{E}) = \mathbb{K}(\mathcal{E}; w)$ of \mathbf{E} by the non-connective K -theory of the Frobenius pair $(\mathbf{Ch}_b(\mathcal{E})_{\text{frob}}, \mathbf{Ch}_b(\mathcal{E})_{\text{frob}}^{qw})$. Then \mathbb{K} is the functor from $\mathbf{RelEx}_{\text{consist}}$ to the stable category of spectra.

Recall the definition of (categorical) homotopy invariant functors from Conventions (6) (iii). We will prove in 3.25 that the non-connective K -theory is a categorical homotopy invariant functor. This result is implicitly utilizing in the proof of Lemma 7 in [Sch06] to do the Eilenberg swindle argument. (See also 7.9 (2)). First recall the following remarks.

Remark 3.23. It is well-known that the functor K^W is a categorical homotopy invariant functor. Namely, if morphisms between Waldhausen exact categories are weakly homotopic, then they induce the same maps on their Waldhausen connective K -theory spectra in the stable category of spectra. (See [Wal85, p.330 1.3.1]). Therefore an exact homotopy equivalence between Waldhausen exact categories is a K^W -equivalence. Namely it induces a homotopy equivalence on Waldhausen K -theory.

Remark 3.24. We recall the definition of the functors \mathfrak{F} and \mathfrak{S} on $\mathbf{BiCompPair}$ from [Sch11, 3.2.23] with slight minor adjustments. Let $\mathbf{C} = (\mathcal{C}, v)$ and $\mathbf{C}' = (\mathcal{C}', v')$ be bicomplial pairs. The argument is carried out in several steps.

- (1) First let us recall the **countable envelope functor** \mathfrak{F} from [Kel90, Appendix B] where $\mathfrak{F}\mathcal{E}$ was written by \mathcal{E}^\sim . We just use the following properties. \mathfrak{F} is a 2-functor from \mathbf{ExCat} to \mathbf{ExCat} which preserves admissible exact sequences of exact functors and there exist a natural transformation $\mathrm{id}_{\mathbf{ExCat}} \rightarrow \mathfrak{F}$.
- (2) Therefore by [Moc10, 2.61], \mathfrak{F} induces the 2-functor $\mathfrak{F} : \mathbf{BiCompPair} \rightarrow \mathbf{BiCompPair}$ and there exists a natural transformation $\xi : \mathrm{id}_{\mathbf{BiCompPair}} \rightarrow \mathfrak{F}$.
- (3) We define the 2-functor $\mathfrak{F} : \mathbf{BiCompPair} \rightarrow \mathbf{BiCompPair}$ by $\mathfrak{F}\mathbf{C} = (\mathfrak{F}\mathcal{C}, w_{\mathfrak{F}\mathbf{C}})$ where a morphism $a : x \rightarrow y$ in $\mathfrak{F}\mathcal{C}$ is in $w_{\mathfrak{F}\mathbf{C}}$ if and only if the image of a by compositions of the canonical projection functors $\mathfrak{F}\mathcal{C} \rightarrow \pi_0(\mathfrak{F}\mathcal{C}) \rightarrow \mathrm{Coker}(\pi_0(\mathfrak{F}(\mathcal{C}^w)) \rightarrow \pi_0(\mathfrak{F}\mathcal{C}))$ is an isomorphism. Then ξ induces a natural transformation $\xi : \mathrm{id}_{\mathbf{BiCompPair}} \rightarrow \mathfrak{F}$.
- (4) By using 3.5 (9), it turns out that \mathfrak{F} preserves relative complicial natural weak equivalences in the following way. Let $q : a \rightarrow b$ be a relative complicial natural weak equivalence between relative complicial functors $a, b : \mathbf{C} \rightarrow \mathbf{C}'$. Then for any object x in \mathcal{C} , Cone_{q_x} is in $\mathcal{C}^{v'}$. Therefore, for any objects x' in $\mathfrak{F}\mathcal{C}$, $\mathrm{Cone}_{\mathfrak{F}q_{x'}}$ is in $\mathfrak{F}(\mathcal{C}^{v'})$. Hence $\mathfrak{F}q$ is in $w_{\mathfrak{F}\mathbf{C}'}$.
- (5) We define the 2-functor $\mathfrak{S} : \mathbf{BiCompPair} \rightarrow \mathbf{BiCompPair}$ by $\mathfrak{S}\mathbf{C} = (\mathfrak{S}\mathcal{C}, w_{\mathfrak{S}\mathbf{C}})$ where a morphism $a : x \rightarrow y$ in $\mathfrak{S}\mathcal{C}$ is in $w_{\mathfrak{S}\mathbf{C}}$ if and only if the image of a by compositions of the canonical projection functors $\mathfrak{S}\mathcal{C} \rightarrow \mathcal{T}(\mathfrak{S}\mathbf{C}) \rightarrow \mathrm{Coker}(\mathcal{T}(\mathbf{C}) \xrightarrow{\mathcal{T}(\xi_{\mathbf{C}})} \mathcal{T}(\mathfrak{S}\mathbf{C}))$ is an isomorphism. By construction and (4), the functor \mathfrak{S} also preserves relative complicial natural weak equivalences.

Corollary 3.25. *The non-connective K -theory is a categorical homotopy invariant functor from $\mathbf{RelEx}_{\mathrm{consist}}$ to the stable category of spectra.*

Proof. For any non-negative integer n , the functor $K^W(\mathfrak{S}^n \mathbf{Ch}_b(-))$ from $\mathbf{RelEx}_{\mathrm{consist}}$ to the stable category of spectra is homotopy invariant by virtue of 2.11 (3), 3.23 and 3.24. Therefore we obtain the result. \square

4 Total quasi-weak equivalences

In this section, let $\mathbf{E} = (\mathcal{E}, w)$ be a bicomplial pair. The main theme in this section is defining and studying the class of *total quasi-weak equivalences* tw on $\mathbf{Ch}_b(\mathcal{E})$. The pivot in this section is the theorem 4.4 and as its by-product, we establish the universal property of $\mathbf{Ch}_b(\mathbf{G})$ for any consistent relative exact category \mathbf{G} in 4.11 and learn that \mathbf{E} is a consistent Waldhausen exact category in 4.13 and as its corollary, we obtain the Gillet-Waldhausen theorem for strict relative exact categories in 4.15.

Remark 4.1. $\mathbf{Ch}_b(\mathcal{E})$ has two bicomplial structure. Namely, the complicial structure induced from \mathcal{E} which is denoted by \mathbf{A} . The second complicial structure is the usual structure on the category of complexes as in 3.2 (1). We denote it by \mathbf{B} . For example $\text{Acy}_b^{lw} \mathcal{E}$ is a null class in \mathbf{A} but is not closed under $C^{\mathbf{B}}$ -contractible objects. $\text{Acy}_b \mathcal{E}$ is a null class in \mathbf{B}_{frob} by 3.15 (2).

Notations 4.2 (Total functor). (cf. [Moc10, 4.15, 4.16]). There exists the triple $(\text{Tot}_{\mathcal{E}}, c^{\mathbf{A}}, c^{\mathbf{B}})$ consisting of an exact functor $\text{Tot}_{\mathcal{E}} = \text{Tot} : \mathbf{Ch}_b(\mathcal{E}) \rightarrow \mathcal{E}$ and natural equivalences $c^{\mathbf{A}} : C^{\mathbf{E}} \text{Tot} \xrightarrow{\sim} \text{Tot } C^{\mathbf{A}}$ and $c^{\mathbf{B}} : C^{\mathbf{B}} \text{Tot} \xrightarrow{\sim} \text{Tot } C^{\mathbf{B}}$ such that both $(\text{Tot}, c^{\mathbf{A}}) : \mathbf{A} \rightarrow \mathbf{E}$ and $(\text{Tot}, c^{\mathbf{B}}) : \mathbf{B} \rightarrow \mathbf{E}$ are complicial functors and $\text{Tot } j_{\mathcal{E}} = \text{id}_{\mathcal{E}}$. It is unique up to the unique complicial natural equivalence. (For more precise statement, please see Ibid). We call it the **total functor** on $\mathbf{Ch}_b(\mathcal{E})$.

Definition 4.3 (Total quasi-weak equivalences). We put $tw := \text{Tot}^{-1} w$ and call it the **class of total quasi-weak equivalences**. We call a morphism in tw a total quasi-weak equivalence. We put $\text{Acy}_b^{tw}(\mathcal{E}) := \mathbf{Ch}_b(\mathcal{E})^{tw}$. Since Tot is a complicial functor, if w is thick, then tw is also thick.

The hinge in this section is the following theorem.

Theorem 4.4. (1) **($tw=lw \vee qis$).** $\text{Acy}_b^{tw} \mathcal{E} = ((\text{Acy}_b^{lw} \mathcal{E})_{qis})_{\text{nul}, \mathbf{B}_{\text{frob}}} = ((\text{Acy}_b \mathcal{E})_{lw})_{\text{nul}, \mathbf{A}}$.
(2) *In particular if the image of $\text{Acy}_b \mathcal{E}$ and $\text{Acy}_b^{lw} \mathcal{E}$ by the projection $\mathbf{A} \rightarrow \pi_0(\mathbf{A})$ (resp. $\mathbf{B} \rightarrow \pi_0(\mathbf{B})$) is factorizable in a suitable order, then we have $\text{Acy}_b^{tw} \mathcal{E} = (\text{Acy}_b \mathcal{E})_{lw}$ (resp. $\text{Acy}_b^{tw} \mathcal{E} = (\text{Acy}_b^{lw} \mathcal{E})_{qis}$).*
(3) $qw = tw_{\text{thi}, \mathbf{B}_{\text{frob}}}$. *In particular if w is thick, $qw = tw$.*

To prove the theorem, we need the several lemmata 4.5, 4.9 and 4.10 below.

Lemma 4.5. *Let (\mathcal{E}, w) be a bicomplial pair. Then $qis \subset tw$.*

Proof. We need only check that for any acyclic complex x in $\mathbf{Ch}_b(\mathcal{E})$, $\text{Tot } x$ is in \mathcal{E}^w . The proof is carried out in several steps.

4.6 (Step 1). Let x be a complex in $\mathbf{Ch}_b(\mathcal{E})$ and let us assume that x is acyclic. Then there exists a strictly acyclic complex y and a $C^{\mathbf{B}}$ -homotopy equivalence $f : x \rightarrow y$. Since Tot preserves C -homotopy equivalences and \mathcal{E}^w is closed under $C^{\mathbf{E}}$ -homotopy equivalences, $\text{Tot } y \in \mathcal{E}^w$ implies $\text{Tot } x \in \mathcal{E}^w$. We shall assume that x is a strictly acyclic complex.

4.7 (Step 2). Since Tot is an exact functor and \mathcal{E}^w is closed under extensions, we shall assume that the length of x is 1 by induction of the length of x and by the admissible exact sequence

$$\tau_{\leq n} x \rightarrowtail x \twoheadrightarrow \tau_{\geq n+1} x.$$

4.8 (Step 3). Since Tot commutes with the suspension functors and \mathcal{E}^w is closed under the suspension functor, we shall assume that $x = [x_1 \xrightarrow{d_1} x_0]$ and d_1 is an isomorphism. In this case, we have the equality $\text{Tot } x = \text{Cone } d_1$ by the construction of Tot and it is in \mathcal{E}^w .

□

Lemma 4.9. *Let $f : x \rightarrow y$ be a morphism in $\mathbf{Ch}_b(\mathcal{E})$. Then*

- (1) *If f is in lw , then $\text{Cone}_{\mathbf{B}} f$ is in $\text{Acy}_b^{lw}(\mathcal{E})_{qis}$. In particular, $lw \subset w_{\text{Acy}_b^{lw}(\mathcal{E})_{qis}}$.*
- (2) *If f is in qis , then $\text{Cone}_{\mathbf{A}} f$ is in $\text{Acy}_b(\mathcal{E})_{lw}$. In particular, $qis \subset w_{\text{Acy}_b(\mathcal{E})_{lw}}$.*

Proof. Let $f : x \rightarrow y$ be a morphism in $\mathbf{Ch}_b(\mathcal{E})$. Assume that f is in lw (resp. qis). Namely $\text{Cone}_A f$ (resp. $\text{Cone}_B f$) is in $\text{Acy}_b^{lw}(\mathcal{E})$ (resp. $\text{Acy}_b(\mathcal{E})$). Let us consider the push out diagram below

$$\begin{array}{ccc} x & \xrightarrow{\quad} & C^\# x \\ f \downarrow & & \downarrow \\ y & \xrightarrow{\quad} & \text{Cone}^\# f \end{array}$$

where $\# = A$ (resp. $\# = B$). Then by [Kel90, p.406 step 1], the morphism $[x \xrightarrow{f} y] \rightarrow [C^\# x \rightarrow \text{Cone}^\# f]$ between the complexes in $\mathbf{Ch}_b(\mathbf{Ch}_b(\mathcal{E}))$ is a quasi-isomorphism. Taking the totalized complex $\text{Tot}^\# : \mathbf{Ch}_b(\mathbf{Ch}_b(\mathcal{E})) \rightarrow \mathbf{Ch}_b(\mathcal{E})$, it turns out that $\text{Cone}^\# f = \text{Tot}^\# [x \xrightarrow{f} y]$ is connected with the complex $\text{Tot}^\# [C^* x \rightarrow \text{Cone}^* \mathcal{E} f]$ in $\text{Acy}_b^{lw}(\mathcal{E})$ (resp. $\text{Acy}_b(\mathcal{E})$) by the morphisms in qis (resp. lw) where $\# = B$ and $*$ = A (resp. $\# = A$ and $*$ = B). Hence we complete the proof. \square

Lemma 4.10. *For any complex x in $\mathbf{Ch}_b(\mathcal{E})$, there is a zig-zag sequence of quasi-isomorphisms and level weak equivalences connecting it to a degree shift of $j_{\mathcal{E}}(\text{Tot } x)$.*

Proof. Let $x = [\cdots \rightarrow 0 \rightarrow x_n \rightarrow x_{n-1} \rightarrow \cdots]$ be a complex in $\mathbf{Ch}_b(\mathcal{E})$. Then we have the following morphisms of complexes.

$$\begin{array}{ccccc} x_n & \xrightarrow{\quad} & C^\mathcal{E}(x_n) & \xleftarrow{\quad} & 0 \\ d_n \downarrow & & \downarrow & & \downarrow \\ x_{n-1} & \xrightarrow{\quad} & \text{Cone}^\mathcal{E} d_{n-1} & \xleftarrow{\text{id}} & \text{Cone}^\mathcal{E} d_{n-1} \\ d_{n-1} \downarrow & & \downarrow & & \downarrow \\ x_{n-2} & \xrightarrow{\quad} & x_{n-2} & \xleftarrow{\text{id}} & x_{n-2} \\ d_{n-2} \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots \end{array}$$

where the left morphism is a quasi-isomorphism by the proof of 4.9 and obviously the right morphism is in lw . Now by induction of the length of x , we give the algorithm of connecting x to a degree shift of $j_{\mathcal{E}}(\text{Tot } x)$ by a zig-zag sequence of quasi-isomorphisms and level-weak equivalences. \square

Proof of Theorem 4.4. To prove assertion (1), we will show the following inclusions.

$$(\text{Acy}_b^{lw} \mathcal{E})_{qis} \subset \text{I} ((\text{Acy}_b \mathcal{E})_{lw})_{nul} \subset \text{II} \text{Acy}_b^{tw} \mathcal{E} \subset \text{III} ((\text{Acy}_b^{lw} \mathcal{E})_{qis})_{nul}$$

The proof for the inclusion **I**: Since the zero complex and any complex x in $\text{Acy}_b^{lw} \mathcal{E}$ is connected by the canonical morphism $0 \rightarrow x$ in lw , we have $\text{Acy}_b^{lw} \mathcal{E} \subset (\text{Acy}_b \mathcal{E})_{lw}$. Now the inclusion **I** follows from 3.5 (10) and 4.9 (2).

The proof for the inclusion **II**: Since $\text{Tot}(\text{Acy}_b^{lw} \mathcal{E}) \subset \mathcal{E}^w$, we have $lw \subset tw$. Therefore the inclusion **II** follows from 3.5 (10) and 4.5.

The proof for the inclusion III: The inclusion III follows from 3.5 (10), 4.9 (1), 4.10 and the inclusion $\text{qis} \subset w_{\text{Acy}_b^{lw}} \mathcal{E}_{\text{qis}}$.

Assertion (2) follows from (1), 1.16 and 3.15 (2). Assertion (3) follows from (1), 2.6 and 3.15 (2). \square

Corollary 4.11 (Universal property of Ch_b). *For any consistent relative exact category $\mathbf{G} = (\mathcal{G}, w_{\mathcal{G}})$ and any thick bicomplial pair $\mathbf{C} = (\mathcal{C}, w_{\mathcal{C}})$ and any relative exact functor $f : \mathbf{G} \rightarrow \mathbf{C}$, there exists a relative complicial functor $(\bar{f}, c) : \text{Ch}_b(\mathbf{G}) \rightarrow \mathbf{C}$ such that $\bar{f}j_{\mathcal{G}} = f$. (\bar{f}, c) is unique in the following sense. For another (\bar{f}', c') such that $\bar{f}'j_{\mathcal{G}} = f$, there exists a unique relative complicial natural equivalence $\theta : \bar{f} \xrightarrow{\sim} \bar{f}'$ such that $\theta j_{\mathcal{G}} = \text{id}_f$.*

Proof. Since \mathbf{C} is thick, the functor $\text{Tot} : \text{Ch}_b(\mathcal{C}) \rightarrow \mathcal{C}$ is a relative exact functor $\text{Ch}_b(\mathbf{C}) \rightarrow \mathbf{C}$ by 4.4 (3). We put $\bar{f} := \text{Tot} \text{Ch}_b(f)$. Then we can easily check that $\bar{f}j_{\mathcal{G}} = f$. Uniqueness of \bar{f} follows from [Moc10, 4.9]. \square

Lemma-Definition 4.12 (Solid axiom). For any strict relative exact category $\mathbf{F} = (\mathcal{F}, v)$, as in the proof of 2.9, we have the implications $(1) \Rightarrow (2) \Rightarrow (3)$ for the conditions below.

- (1) $lv \subset v_{\text{Acy}_b^{lv}(\mathcal{F})_{\text{qis}}}$.
- (2) For any morphism $f : x \rightarrow y$ in v , there is a zig-zag sequence of quasi-isomorphisms connecting the complex $\text{Cone } f = [x \xrightarrow{f} y]$ to a bounded complex in \mathcal{F}^v .
- (3) $lv \subset v_{(\text{Acy}_b^{lv}(\mathcal{F})_{\text{qis}})_{\text{mnl}}}$.

We say that v or \mathbf{F} satisfies the **solid axiom** or v or \mathbf{F} is solid if v satisfies condition (2) above. We can easily check that the solid axiom implies the consistent axiom. We denote the full 2-subcategory of solid relative exact categories (resp. solid Waldhausen exact categories) in $\text{RelEx}^{\#}$ by $\text{RelEx}_{\text{solid}}^{\#}$ (resp. $\text{WalEx}_{\text{solid}}^{\#}$) for $\# \in \{+, \text{nothing}\}$. \square

Corollary 4.13. *For any bicomplial pair (\mathcal{F}, v) , v satisfies the solid axiom. In particular, for any strict relative exact category $\mathbf{G} = (\mathcal{G}, u)$, $\text{Ch}_b(\mathbf{G}) = (\text{Ch}_b(\mathcal{G}), qu)$ is a solid Waldhausen exact category.* \square

Corollary 4.14. *For any morphism $f : x \rightarrow y$ in $\text{Ch}_b(\mathcal{E})$, $\text{Cone}^{\mathbf{A}} f$ is canonically quasi-weak equivalent to $\text{Cone}^{\mathbf{B}} f$.*

Proof. The canonical morphism $[0 \rightarrow \text{Cone}^{\mathbf{A}} f] \rightarrow [C^{\mathbf{A}} x \rightarrow \text{Cone}^{\mathbf{A}} f]$ in $\text{Ch}_b(\text{Ch}_b(\mathcal{E}))$ is in llw . Therefore by taking the total functor, we have the zig-zag sequence of the quasi-isomorphism and the level weak equivalence

$$\text{Cone}^{\mathbf{B}} f \rightarrow \text{Tot}[C^{\mathbf{A}} x \rightarrow \text{Cone}^{\mathbf{A}} f] \leftarrow \text{Cone}^{\mathbf{A}} f.$$

Since a level weak equivalence is a quasi weak equivalence by 4.4 and 4.9 (1), we get the desired result. \square

Corollary 4.15. (1) *Let us assume that \mathbf{E} is thick. Then the canonical functor $\mathbf{E} \xrightarrow{j_{\mathcal{E}}} \text{Ch}_b(\mathbf{E})$ induces an equivalence of triangulated categories*

$$\mathcal{T}(\mathcal{E}, w) \xrightarrow{\sim} \mathcal{T}(\text{Ch}_b(\mathcal{E}), qw) = \mathcal{D}(\mathcal{E}, w).$$

(2) **(Derived Gillet-Waldhausen theorem).** *Let $\mathbf{F} = (\mathcal{F}, v)$ be a strict relative exact category, then the canonical inclusion functor $j_{\mathcal{F}} : \mathbf{F} \rightarrow \text{Ch}_b(\mathbf{F})$ is a derived equivalence.*

Proof. (1) First, we will prove that $\text{Tot} : \mathbf{Ch}_b(\mathcal{E}) \rightarrow \mathcal{E}$ induces an equivalence of triangulated categories

$$\mathcal{T}(\text{Tot}) : \mathcal{T}(\mathbf{Ch}_b(\mathcal{E}), tw) \xrightarrow{\sim} \mathcal{T}(\mathcal{E}, w).$$

Since $\pi_0(\mathcal{E}^w)$ is thick by 3.5 (2), we have the equality

$$\pi_0(\text{Acy}_b^{tw}(\mathcal{E})) = \text{Ker}(\pi_0(\mathbf{Ch}_b(\mathcal{E})) \xrightarrow{\pi_0(\text{Tot})} \pi_0(\mathcal{E}) \rightarrow \mathcal{T}(\mathcal{E}, w)).$$

Therefore for any object x in $\mathcal{T}(\mathbf{Ch}_b(\mathcal{E}), tw)$, $\mathcal{T}(\text{Tot})(x) \xrightarrow{\sim} 0$ implies $x \xrightarrow{\sim} 0$ in $\mathcal{T}(\mathbf{Ch}_b(\mathcal{E}), tw)$. Since we have $\text{Tot } j_{\mathcal{E}} = \text{id}$, $\mathcal{T}(\text{Tot})$ is essentially surjective. Moreover by 4.4 and 4.10, for any complexes x and y in $\mathbf{Ch}_b(\mathcal{E})$, there exist isomorphisms $a : j_{\mathcal{E}}(\text{Tot } x) \xrightarrow{\sim} x$ and $y \xrightarrow{\sim} j_{\mathcal{E}}(\text{Tot } y)$ in $\mathcal{T}(\mathbf{Ch}_b(\mathcal{E}), tw)$. Now let us consider the commutative diagram below.

$$\begin{array}{ccc} \text{Hom}_{\mathcal{T}(\mathbf{Ch}_b(\mathcal{E}), tw)}(x, y) & \xrightarrow{\mathcal{T}(\text{Tot})} & \text{Hom}_{\mathcal{T}(\mathcal{E}; w)}(\text{Tot } x, \text{Tot } y) \\ \text{Hom}(a, b) \downarrow \wr & & \wr \downarrow \text{Hom}(\mathcal{T}(\text{Tot})(a), \mathcal{T}(\text{Tot})(b)) \\ \text{Hom}_{\mathcal{T}(\mathbf{Ch}_b(\mathcal{E}), tw)}(j_{\mathcal{E}}(x), j_{\mathcal{E}}(y)) & \xrightarrow{\mathcal{T}(\text{Tot})} & \text{Hom}_{\mathcal{T}(\mathcal{E}; w)}(\text{Tot } x, \text{Tot } y). \end{array}$$

Since the bottom $\mathcal{T}(\text{Tot})$ has the section $j_{\mathcal{E}}$, it is surjective and therefore the top $\mathcal{T}(\text{Tot})$ is also. Hence $\mathcal{T}(\text{Tot})$ is full. Now utilizing [Bal07, 3.18], it turns out that $\mathcal{T}(\text{Tot})$ is an equivalence of triangulated categories. Since $\text{Tot } j_{\mathcal{E}} = \text{id}$, $j_{\mathcal{E}}$ is the inverse functor of the equivalence above. By 3.16 and 4.4 (3), the identity functor $(\mathbf{Ch}_b(\mathcal{E}), tw) \rightarrow (\mathbf{Ch}_b(\mathcal{E}), qw)$ induces an equivalence of triangulated categories $\mathcal{T}(\mathbf{Ch}_b(\mathcal{E}), tw) \xrightarrow{\sim} \mathcal{T}(\mathbf{Ch}_b(\mathcal{E}), qw)$. Hence we obtain the result.

(2) By 3.20 and 4.13, $\mathbf{Ch}_b(\mathbf{F})$ is a solid thick bicomplial pair. Then by applying assertion (1) to $\mathbf{E} = \mathbf{Ch}_b(\mathbf{F})$, we obtain an equivalence of triangulated categories $\mathcal{D}_b(\mathbf{F}) = \mathcal{T}(\mathbf{Ch}_b(\mathbf{F}), qu) \xrightarrow{\sim} \mathcal{D}_b(\mathbf{Ch}_b(\mathbf{F}))$. \square

Recall the terminologies in relative category theory from Conventions (6).

Corollary 4.16. *Let U be the forgetful 2-functor from $\mathbf{BiCompPair}_{\text{thi}}^+$ to $\mathbf{RelEx}_{\text{consist}}^+$. Then the pair $(\mathbf{RelEx}_{\text{consist}}^+, \text{deq}) \xrightarrow[\text{U}]{\text{Ch}_b} (\mathbf{BiCompPair}_{\text{thi}}^+, \text{deq})$ are relative functors with adjunction morphisms $j; \text{id}_{\mathbf{RelEx}_{\text{consist}}^+} \rightarrow U \mathbf{Ch}_b$ and $\mathbf{Ch}_b U \rightarrow \text{id}_{\mathbf{BiCompPair}_{\text{thi}}^+}$. Moreover j and Tot are relative natural equivalences. In particular the homotopy theories of consistent relative exact categories and thick bicomplial pairs are homotopy equivalent.* \square

Recall the definition of exact and weakly exact functors from 2.7 and the definition of quotient of strictly relative exact categories from 3.17.

Corollary 4.17. (1) *The functor $\mathbf{Ch}_b : \mathbf{RelEx}_{\text{consist}} \rightarrow \mathbf{RelEx}_{\text{consist}}$ is exact and weakly exact.*
 (2) *Let $\mathbf{G} = (\mathcal{G}, u)$ be a very strict solid relative exact category. Then*
 (i) *The inclusion functor $\mathbf{Ch}_b(\mathcal{G}^u) \hookrightarrow \mathbf{Ch}_b(\mathcal{G})^{qu}$ induces an equivalence of triangulated categories $\mathcal{D}_b(\mathcal{G}^u) \xrightarrow{\sim} \mathcal{T}(\mathbf{Ch}_b(\mathcal{G})^{qu}, \text{qis})$.*
 (ii) *The inclusion functor $\mathcal{G}^u \hookrightarrow \mathcal{G}$ and the identity functor of \mathcal{G} induce an exact sequence*

$$(\mathcal{G}^u, i_{\mathcal{G}^u}) \rightarrow (\mathcal{G}, i_{\mathcal{G}}) \rightarrow (\mathcal{G}, u).$$

(3) *Let $f : \mathbf{F} = (\mathcal{F}, v) \rightarrow \mathbf{G} = (\mathcal{G}, u)$ be a derived fully faithful relative exact functor from a strict relative exact category \mathbf{F} to a consistent relative exact category \mathbf{G} . Then the sequence*

$\mathbf{F} \xrightarrow{f} \mathbf{G} \xrightarrow{\pi_f} \mathbf{G} / \mathbf{F}$ is weakly exact.
(4) Let

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{f} & \mathbf{G} = (\mathcal{G}, u) \\ \downarrow a & & \downarrow b \\ \mathbf{F}' & \xrightarrow{f'} & \mathbf{G}' = (\mathcal{G}', u') \end{array}$$

be a commutative diagram of strict relative exact categories. if both a and b are derived equivalences, then $\text{Coker}(a, b)$ is also a derived equivalence.

Proof. (1) Let $\mathbf{G} = (\mathcal{G}, w_{\mathcal{G}}) \xrightarrow{a} \mathbf{H} = (\mathcal{H}, w_{\mathcal{H}}) \xrightarrow{b} \mathbf{I} = (\mathcal{I}, w_{\mathcal{I}})$ be a sequence of consistent relative exact categories. Consider the commutative diagram of triangulated categories

$$\begin{array}{ccccc} \mathcal{D}_b(\mathbf{G}) & \xrightarrow{\mathcal{D}_b(a)} & \mathcal{D}_b(\mathbf{H}) & \xrightarrow{\mathcal{D}_b(b)} & \mathcal{D}_b(\mathbf{I}) \\ \downarrow \mathcal{D}_b(j_{\mathcal{G}}) & & \downarrow \mathcal{D}_b(j_{\mathcal{H}}) & & \downarrow \mathcal{D}_b(j_{\mathcal{I}}) \\ \mathcal{D}_b(\text{Ch}_b(\mathbf{G})) & \xrightarrow{\mathcal{D}_b(\text{Ch}_b(a))} & \mathcal{D}_b(\text{Ch}_b(\mathbf{H})) & \xrightarrow{\mathcal{D}_b(\text{Ch}_b(b))} & \mathcal{D}_b(\text{Ch}_b(\mathbf{I})). \end{array}$$

Here the vertical morphisms are equivalences of triangulated categories by 4.15 (2). Hence if the top line is exact (resp. weakly exact), then the bottom line is also exact (resp. weakly exact).

(2) Let us consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{D}_b(\mathcal{G}^u) & \xrightarrow{\text{I}} & \mathcal{D}_b(\mathcal{G}) & \xrightarrow{\quad} & \mathcal{D}_b(\mathbf{G}) \\ \text{III} \downarrow & & \parallel & & \parallel \\ \mathcal{T}(\text{Ch}_b(\mathcal{G})^{qu}, \text{qis}) & \xrightarrow{\text{II}} & \mathcal{T}(\text{Ch}_b(\mathcal{G}), \text{qis}) & \xrightarrow{\quad} & \mathcal{T}(\text{Ch}_b(\mathcal{G}), qu) \end{array} \quad (2)$$

The functors **I** and **II** are fully faithful by the assumption and 3.20 (2) respectively. Therefore the functor **III** is also fully faithful. Since \mathbf{G} is solid, we have an equality $\text{Ch}_b(\mathcal{G})^{qu} = \text{Ch}_b(\mathcal{G}^u)_{\text{qis}}$. Therefore the functor **III** is essentially surjective. Hence we complete the proof of (i). Now the exactness of the bottom line in the diagram (2) implies the exactness of the top line. We obtain the proof of (ii).

(3) Consider the following commutative diagram of triangulated categories.

$$\begin{array}{ccccc} \mathcal{D}_b(\mathbf{F}) & \xrightarrow{\mathcal{D}_b(f)} & \mathcal{D}_b(\mathbf{G}) & \xrightarrow{\mathcal{T}(\text{id}_{\text{Ch}_b(\mathcal{G})})} & \mathcal{T}(\text{Ch}_b(\mathcal{G}), w_f) \\ \parallel & & \parallel & & \downarrow \mathcal{T}(j_{\text{Ch}_b(\mathcal{G})}) \\ \mathcal{D}_b(\mathbf{F}) & \xrightarrow{\mathcal{D}_b(f)} & \mathcal{D}_b(\mathbf{G}) & \xrightarrow{\mathcal{D}_b(\pi_f)} & \mathcal{D}_b(\mathbf{G} / \mathbf{F}). \end{array}$$

Here the top line is weakly exact and the right vertical line is an equivalences of triangulated categories by 4.15 (2). Hence we obtain the result.

(4) Consider the following commutative diagram

$$\begin{array}{ccccc}
\text{Coker } \mathcal{D}_b(f) & \xrightarrow[\sim]{\text{I}} & \mathcal{T}(\text{Ch}_b(\mathcal{G}), w_f) & \xrightarrow[\sim]{\text{III}} & \mathcal{D}_b(\mathbf{G} / \mathbf{F}) \\
\downarrow \text{II} & & \downarrow \mathcal{T}(\text{Ch}_b(b)) & & \downarrow \mathcal{D}_b(\text{Coker}(a, b)) \\
\text{Coker } \mathcal{D}_b(f') & \xrightarrow[\text{I}]{\sim} & \mathcal{T}(\text{Ch}_b(\mathcal{G}'), w_{f'}) & \xrightarrow[\text{III}]{\sim} & \mathcal{D}_b(\mathbf{G}' / \mathbf{F}').
\end{array}$$

Here the morphisms **I**, **II** and **III** are equivalences of triangulated categories by 3.18 (1), assumption and 4.15 (2) respectively. Hence the relative exact functor $\text{Coker}(a, b)$ is a derived equivalence. \square

4.18 (Proof of Corollary 0.2). For any weakly exact sequence of consistent relative exact categories $\mathbf{E} \xrightarrow{u} \mathbf{F} \xrightarrow{v} \mathbf{G}$, the induced sequence $\text{Ch}_b(\mathbf{E}) \xrightarrow{\text{Ch}_b(u)} \text{Ch}_b(\mathbf{F}) \xrightarrow{\text{Ch}_b(v)} \text{Ch}_b(\mathbf{G})$ is a weakly exact sequence of bicomplial pairs and complial exact functors by 3.20 (1) and 4.17 (1). Therefore the assertion follows from [Sch06, 3.2.27]. \square

5 Resolution theorems

In this section, we review the (strongly) resolution conditions in 5.2 and introduce the resolution theorems in 5.3. Recall the definition of multiplicative systems from Conventions (5) (xiv).

5.1. For a pair of a category \mathcal{C} and a multiplicative system v of \mathcal{C} , we define the simplicial subcategory $\mathcal{C}(-, v)$ in $[m] \mapsto \mathcal{C}^{[m]}$

$$[m] \mapsto \mathcal{C}(m, v)$$

where $\mathcal{C}(m, v)$ is the full subcategory of $\mathcal{C}^{[m]}$ consisting of those functors which take values in v .

Definition 5.2 (Resolution conditions). (1) We say that the inclusion functor of Quillen exact categories $\mathcal{E} \hookrightarrow \mathcal{F}$ satisfies **the resolution conditions** if it satisfies the following three conditions.

(Res 1) \mathcal{E} is closed under extensions in \mathcal{F} .

(Res 2) For any object x in \mathcal{F} , there are an object y in \mathcal{E} and an admissible epimorphism $y \twoheadrightarrow x$.

(Res 3) For any admissible short exact sequences $x \rightarrowtail y \twoheadrightarrow z$ in \mathcal{F} , if y is in \mathcal{E} , then x is also in \mathcal{E} .

(2) (cf. [Moc13, 1.12]). Moreover assume that there exists a class of morphisms v in \mathcal{F} such that the pair (\mathcal{F}, v) is a Waldhausen exact category. Let us put $w = \mathcal{E} \cap v$. We say that the inclusion functor $(\mathcal{E}, w) \hookrightarrow (\mathcal{F}, v)$ satisfies **the strongly resolution conditions** if for any non-negative integer m , the inclusion functor $\mathcal{E}(m, w) \hookrightarrow \mathcal{F}(m, v)$ satisfies the resolution conditions.

Theorem 5.3. Let $i : \mathbf{E} = (\mathcal{E}, w) \hookrightarrow \mathbf{F} = (\mathcal{F}, v)$ be an inclusion functor between strict relative exact categories.

(1) **(Derived resolution theorem).** Let us assume that both the inclusion functors $\mathcal{E} \hookrightarrow \mathcal{F}$ and $\mathcal{E}^w \hookrightarrow \mathcal{F}^v$ satisfy the resolution conditions. Then i is a derived equivalence.

(2) **(Resolution theorem).** Let us assume that the following conditions hold.

(i) $v \cap \mathcal{E} = w$.

(ii) Both \mathbf{E} and \mathbf{F} are Waldhausen exact categories.

(iii) $(\mathcal{E}, w) \hookrightarrow (\mathcal{F}, v)$ satisfies the strongly resolution conditions.

Then i is a K^W -equivalence.

Proof. (1) If the classes v and w are the class of all isomorphisms, then the assertion is proven in [Sch11, 3.3.8]. Therefore for general cases, by the assumption, the inclusion functors $\mathcal{E} \hookrightarrow \mathcal{F}$ and $\mathcal{E}^w \hookrightarrow \mathcal{F}^v$ induce equivalences of triangulated categories

$$\mathcal{D}_b(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}_b(\mathcal{F}), \quad \mathcal{D}_b(\mathcal{E}^w) \xrightarrow{\sim} \mathcal{D}_b(\mathcal{F}^v).$$

Hence the inclusion functor $(\mathcal{E}, w) \hookrightarrow (\mathcal{F}, v)$ induces an equivalence of triangulated categories

$$\mathcal{D}_b(\mathcal{E}, w) \xrightarrow{\sim} \mathcal{D}_b(\mathcal{F}, v).$$

Assertion (2) is proven in [Moc13, §1]. □

Corollary 5.4. *Let $i : \mathbf{E} = (\mathcal{E}, w) \hookrightarrow \mathbf{F} = (\mathcal{F}, v)$ be an inclusion functor between strict relative exact categories. Assume that the following conditions hold.*

- (1) $\mathcal{E}^w \hookrightarrow \mathcal{F}^v$ satisfies the resolution conditions.
- (2) v satisfies the solid axiom.

Then w also satisfies the solid axiom.

Proof. Let $f : x \rightarrow y$ be a morphism in w . Then by assumption (2), the complex $\text{Cone } f = [x \xrightarrow{f} y]$ is quasi isomorphic to a complex in $\text{Ch}_b(\mathcal{F}^v)$. Now by the assumption (1), $(\mathcal{E}^w, i_{\mathcal{E}^w}) \rightarrow (\mathcal{F}^v, i_{\mathcal{F}^v})$ is a derived equivalence by 5.3 (1). Therefore the complex $\text{Cone } f$ is quasi-isomorphic to a complex in $\text{Ch}_b(\mathcal{E}^w)$. □

6 Quasi-split exact sequences

In this section, we will prepare the terminologies about quasi-split exact sequences and flags of triangulated categories or particular relative exact categories. We start by recollecting a profitable lemma to manage adjoint functors. Recall the definition of triangle adjoint from Convention (8) (vii).

Lemma 6.1. (1) *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a functor between categories. Assume that f admits a right (resp. left) adjoint functor $g : \mathcal{Y} \rightarrow \mathcal{X}$ with adjunction maps $fg \xrightarrow{A} \text{id}_{\mathcal{Y}}$ and $\text{id}_{\mathcal{X}} \xrightarrow{B} gf$ (resp. $\text{id}_{\mathcal{Y}} \xrightarrow{A} fg$ and $gf \xrightarrow{B} \text{id}_{\mathcal{X}}$) and assume that B is a natural equivalence. Then f is fully faithful.*
(2) *Let (\mathcal{T}, Σ) and (\mathcal{T}', Σ') be triangulated categories, $(g, \rho) : \mathcal{T}' \rightarrow \mathcal{T}$ a triangle functor, f a left (resp. right) adjoint of g with $\Phi : fg \rightarrow \text{id}_{\mathcal{T}}$ and $\Psi : \text{id}_{\mathcal{T}'} \rightarrow gf$ (resp. $\Phi : \text{id}_{\mathcal{T}} \rightarrow fg$ and $\Psi : gf \rightarrow \text{id}_{\mathcal{T}'}$) adjunction morphisms and $\lambda := (\Phi \Sigma' L)(L \rho^{-1} L)(L \Sigma \Psi)$ (resp. $\lambda := \{(f \Sigma \Psi)(f \rho f)(\Phi \Sigma' f)\}^{-1}$). Then (L, λ) is a triangle functor and it is a left (resp. right) triangle adjoint of (R, ρ) .*

Proof. Assertion (2) for left adjoint case is mentioned in [Kel96, 8.3]. We will prove assertion (1). Since f has a left quasi-inverse functor g , f is faithful. We will prove that f is full. For any morphism $a : f(x) \rightarrow f(y)$ in \mathcal{Y} , we put $b := (By)^{-1}gaBx$ (resp. $b := Byga(Bx)^{-1}$). Then we have the equality

$$fb = (fBy)^{-1}fgafBx = (fBy)^{-1}(Afy)^{-1}aAfxfBx = a$$

$$\text{(resp. } fByfga(fBx)^{-1} = fByAfyga(Afx)^{-1}(fBx)^{-1} = a).$$

Hence f is full. □

Definition 6.2 (Relative exact adjoint functors). Let $f : \mathbf{E} = (\mathcal{E}, w) \rightarrow \mathbf{E}' = (\mathcal{E}', w')$ and $g : \mathbf{E}' \rightarrow \mathbf{E}$ be relative exact functors between relative exact categories and $A : \text{id}_{\mathcal{E}} \rightarrow gf$ and $B : fg \rightarrow \text{id}_{\mathcal{E}'}$ natural transformations such that $(Bf)(fA) = \text{id}_f$ and $(gB)(Ag) = \text{id}_g$. Then we say that f (resp. g) is a **left** (resp. **right**) relative exact adjoint functor of g (resp. f).

Lemma-Definition 6.3 (Quasi-split exact sequences). (1) Let

$$\mathbf{E} = (\mathcal{E}, w) \xrightarrow[i]{i} \mathbf{F} = (\mathcal{F}, v) \xrightarrow[j]{p} \mathbf{G} = (\mathcal{G}, u)$$

be relative exact functors between relative exact categories and $A : iq \rightarrow \text{id}_{\mathcal{F}}$, $B : \text{id}_{\mathcal{E}} \rightarrow qi$, $C : \text{id}_{\mathcal{F}} \rightarrow jp$ and $D : pj \rightarrow \text{id}_{\mathcal{G}}$ are natural transformations such that $(Ai)(iB) = \text{id}_i$, $(qA)(Bq) = \text{id}_q$, $(jD)(Cj) = \text{id}_j$ and $(Dp)(pC) = \text{id}_p$. Suppose that a sequence $iq \xrightarrow{A} \text{id}_{\mathcal{F}} \xrightarrow{C} jp$ is admissible exact. Then

- (i) The natural transformations Ai , iB , Cj and jD are natural equivalences.
- (ii) The bifunctor $\text{Hom}_{\mathcal{F}}(i(-), j(-))$ from $\mathcal{E}^{\text{op}} \times \mathcal{G}$ to the category of abelian groups is trivial.
- (iii) The functors pi and qj are trivial.
- (iv) The following conditions are equivalent.
 - (a) i (resp. j) is fully faithful.
 - (b) i (resp. j) is conservative.
 - (c) B (resp. D) is a natural equivalence.
- (v) If the conditions in (iv) is verified, then the functor i (resp. j) reflects exactness and the functor pi (resp. qj) is trivial.

If the equivalent conditions in (iv) hold, namely both i and j are fully faithful and both B and D are natural equivalences, then we call the sequence $\mathbf{E} \xrightarrow{i} \mathbf{F} \xrightarrow{p} \mathbf{G}$ (resp. $\mathbf{G} \xrightarrow{j} \mathbf{F} \xrightarrow{q} \mathbf{E}$) a **right** (resp. **left**) **quasi-split exact sequence**. We say that a system (j, q, A, B, C, D) (resp. (i, p, C, D, A, B)) or shortyl (j, q) (resp. (i, p)) is a **right** (resp. **left**) **quasi-splitting** of a sequence (i, p) (resp. (j, q)).

(2) Let $(\mathcal{T}, \Sigma) \xrightarrow{i} (\mathcal{T}', \Sigma') \xrightarrow{p} (\mathcal{T}'', \Sigma'')$ be a sequence of triangulated categories. We say that a sequence (i, p) is **right** (resp. **left**) **quasi-split** if both i and p admit right (resp. left) adjoint functors $q : \mathcal{T}' \rightarrow \mathcal{T}$ and $j : \mathcal{T}'' \rightarrow \mathcal{T}'$ with adjunction maps $iq \xrightarrow{A} \text{id}_{\mathcal{T}'}$, $\text{id}_{\mathcal{T}} \xrightarrow{B} qi$, $\text{id}_{\mathcal{T}'} \xrightarrow{C} jp$ and $pj \xrightarrow{D} \text{id}_{\mathcal{T}''}$ (resp. $\text{id}_{\mathcal{T}'} \xrightarrow{A} iq$, $qi \xrightarrow{B} \text{id}_{\mathcal{T}}$, $jp \xrightarrow{C} \text{id}_{\mathcal{T}'}$ and $\text{id}_{\mathcal{T}''} \xrightarrow{D} pj$) respectively such that B and D are natural equivalences and if there exists a triangle natural transformation $jp \xrightarrow{E} \Sigma' iq$ (resp. $iq \xrightarrow{E} \Sigma' jp$) such that a triangle (A, C, E) (resp. (C, A, E)) is a Σ' -exact triangle. We call a system (j, q, A, B, C, D, E) (resp. (j, q, C, D, A, B, E)) or shortly (j, q, E) a **right** (resp. **left**) **splitting** of a sequence (i, p) . Then

- (i) The functors i and j are fully faithful.
- (ii) The natural transformations Ai and Cj are natural equivalences.
- (iii) The functors pi and qj are trivial.
- (iv) The bifunctor $\text{Hom}_{\mathcal{T}'}(i(-), j(-))$ (resp. $\text{Hom}_{\mathcal{T}'}(j(-), i(-))$) from $\mathcal{T}^{\text{op}} \times \mathcal{T}''$ (resp. $\mathcal{T}''^{\text{op}} \times \mathcal{T}$) to the category of abelian groups is trivial.

(3) A sequence of strict relative exact categories $\mathbf{E} \xrightarrow{i} \mathbf{F} \xrightarrow{p} \mathbf{G}$ is a **right** (resp. **left**) **quasi-split exact sequence** if the induced sequence $\mathcal{D}_b(\mathbf{E}) \xrightarrow{\mathcal{D}_b(i)} \mathcal{D}_b(\mathbf{F}) \xrightarrow{\mathcal{D}_b(p)} \mathcal{D}_b(\mathbf{G})$ is a right (resp. left) quasi-split exact sequence of triangulated categories.

Proof. (1) (i) By exactness $iq \xrightarrow{A} \text{id}_{\mathcal{F}} \xrightarrow{C} jp$, Aix (resp. Cjx) is a monomorphism (resp. an epimorphism) for any object x in \mathcal{E} (resp. \mathcal{G}). On the other hand, Aix (resp. Cjx) has a right (resp. left) inverse iBx (resp. jDx). Therefore Aix (resp. Cjx) and its right (resp. left) inverser iBx (resp. jDx) are isomorphisms.

(ii) In the exact sequence $iqi \xrightarrow{Ai} i \xrightarrow{Ci} jpi$ (resp. $iqj \xrightarrow{Aj} j \xrightarrow{Cj} j pj$), since Ai (resp. Cj) is an

isomorphism, we have $jpi \xrightarrow{\sim} 0$ (resp. $iqj \xrightarrow{\sim} 0$). Then, for any objects x in \mathcal{E} and z in \mathcal{G} and any morphism $a : ix \rightarrow jz$, we have equalities

$$a = (Cjz)^{-1}(Cjz)a = (Cjz)^{-1}(jpa)(Cix) = 0.$$

$$\begin{array}{ccccc} ix & \xrightarrow{Cix} & jpix & \xrightarrow{\sim} & 0 \\ a \downarrow & & \downarrow jpa & & \\ jz & \xrightarrow{Cjz} & jpjz & & \end{array}$$

(iii) By assumption, we have the isomorphisms of bifunctors

$$H_{\mathcal{T}'}(i(-), j(-)) \xrightarrow{\sim} \text{Hom}_{\mathcal{T}''}(-, qj(-)) \xrightarrow{\sim} \text{Hom}_{\mathcal{T}}(pi(-), -)$$

and they are trivial by (ii). Hence we obtain the result.

(iv) Implications (a) \Rightarrow (b) \Rightarrow (c) are straightfoward. By 6.1 (1), assertion (c) implies assertion (a).

(v) Let $x \xrightarrow{a} y \xrightarrow{b} z$ be a sequence in \mathcal{E} and assume that $ix \xrightarrow{ia} iy \xrightarrow{ib} iz$ is an admissible exact sequence in \mathcal{F} . Consider the commutative diagram below.

$$\begin{array}{ccccc} x & \xrightarrow{a} & y & \xrightarrow{b} & z \\ Bx \downarrow \wr & & By \downarrow \wr & & \wr \downarrow Bz \\ qix & \xrightarrow{qia} & qiy & \xrightarrow{qib} & qiz. \end{array}$$

Here the bottom line is an admissible exact sequence and the vertical lines are isomorphisms. Hence a sequence (a, b) is an admissible exact sequence in \mathcal{E} . The other assertions are trivial. We can similarly prove the assertions in (2). \square

Remark 6.4. For a right (resp. left) quasi-split exact sequence of triangulated categoriees, we can take a right (resp. left) quasi-splitting of it as a system of triangulated adjoint functors and triangle natural transformations by virtue of 6.1 (2).

Recall the definition of the category of admissible exact sequences in an exact category from Conventions (7) (xiv) and 2.11 (1).

Corollary 6.5. Let $\mathbf{E} = (\mathcal{E}, w) \xrightarrow{i} \mathbf{F} = (\mathcal{F}, v) \xrightarrow{p} \mathbf{G} = (\mathcal{G}, u)$ be a right quasi-split exact sequence of relative exact categories with a right quasi-splitting (j, q, A, B, C, D) . Then

(1) For any admissible exact sequence in \mathcal{F} , $ix \xrightarrow{a} y \xrightarrow{b} jz$ with x in \mathcal{E} , y in \mathcal{F} and z in \mathcal{G} , there are unique isomorphisms $\alpha : ix \xrightarrow{\sim} iqy$ and $\beta : jz \xrightarrow{\sim} jpy$ which make the diagram below commutative.

$$\begin{array}{ccccc} ix & \xrightarrow{a} & y & \xrightarrow{c} & jz \\ \alpha \downarrow \wr & & \parallel & & \wr \downarrow \beta \\ iqy & \xrightarrow{Ay} & y & \xrightarrow{Cy} & jpy. \end{array}$$

(2) We regard \mathcal{E} and \mathcal{G} as strict exact subcategories of \mathcal{F} by the exact functors i and j respectively. The functor $\Theta : \mathcal{F} \rightarrow E(\mathcal{E}, \mathcal{F}, \mathcal{G})$ which sends an object y in \mathcal{F} to an admissible exact sequence $iqy \xrightarrow{Ay} y \xrightarrow{Cy} jpy$ gives an equivalence of relative exact categories $\mathbf{F} \rightarrow E(\mathbf{E}, \mathbf{F}, \mathbf{G})$ which makes the diagrams below commutative up to unique natural equivalences.

$$\begin{array}{ccccc}
 \mathbf{E} & \xrightarrow{i} & \mathbf{F} & \xrightarrow{p} & \mathbf{G} & & \mathbf{G} & \xrightarrow{j} & \mathbf{F} & \xrightarrow{q} & \mathbf{E} \\
 \parallel & & \downarrow \Theta & & \parallel & & \parallel & & \downarrow \Theta & & \parallel \\
 \mathbf{E} & \xrightarrow{\text{In}_{\mathcal{E}}} & E(\mathbf{E}, \mathbf{F}, \mathbf{G}) & \xrightarrow{q_{E(\mathcal{E}, \mathcal{F}, \mathcal{G})}} & \mathbf{G}, & & \mathbf{G} & \xrightarrow{\text{In}_{\mathcal{G}}} & E(\mathbf{E}, \mathbf{F}, \mathbf{G}) & \xrightarrow{s_{E(\mathcal{E}, \mathcal{F}, \mathcal{G})}} & \mathbf{E}.
 \end{array}$$

Proof. (1) Since the compositions Cya and cAy are trivial by 6.3 (1) (iii) we have the morphisms $\alpha : ix \rightarrow iqy$, $\beta : iz \rightarrow jpy$, $\alpha' : iqy \rightarrow ix$ and $\beta' : jpy \rightarrow iz$ such that $Ay\alpha = a$, $\beta c = Cy$, $\alpha\alpha' = Ay$ and $\beta'\beta = c$. Since a and Ay are monomorphisms and c and Cy are epimorphisms, α , α' , β and β' are unique. By uniqueness, it turns out that α and α' , β and β' are inverse morphisms in each others respectively.

(2) The inverse functor of Θ is given by $m = m_{E(\mathcal{E}, \mathcal{F}, \mathcal{G})}$. $m\Theta = \text{id}_{\mathcal{F}}$ is trivial. The unique natural equivalence $\Omega : \Theta m \rightarrow \text{id}_{E(\mathcal{E}, \mathcal{F}, \mathcal{G})}$ such that $m\Omega = \text{id}_{\text{id}_{\mathcal{F}}}$ is given by assertion (1). \square

Corollary 6.6. (1) A right (resp. left) quasi-split exact sequence of triangulated categories is exact.

(2) A derived right (resp. left) quasi-split exact sequence of strict relative exact categories is exact.

Proof. (1) We will only give a proof for a right quasi-split case. Let $(\mathcal{T}, \Sigma) \xrightarrow{i} (\mathcal{T}', \Sigma') \xrightarrow{p} (\mathcal{T}'', \Sigma'')$ be a right quasi-split exact sequence with a right quasi-splitting (j, q, A, B, C, D, E) . We write same letters i, q, p and j for the induced functors $\mathcal{T} \xrightarrow{p} \text{Ker } p$, $\mathcal{T}' / \mathcal{T} \xrightarrow{q} \mathcal{T}''$. By assumption, B and D gives equivalences of functors $\text{id}_{\mathcal{T}} \xrightarrow{q} qi$ and $pj \xrightarrow{q} \text{id}_{\mathcal{T}''}$ respectively. Since jp (resp. iq) is trivial on $\text{Ker } p$ (resp. $\mathcal{T}' / \mathcal{T}$), it turns out that A (resp. C) gives an equivalence of functors $iq \xrightarrow{q} \text{id}_{\text{Ker } p}$ (resp. $\text{id}_{\mathcal{T}' / \mathcal{T}} \xrightarrow{q} jp$) by the Σ' -exact triangle $iq \xrightarrow{A} \text{id}_{\mathcal{T}'} \xrightarrow{C} jp \xrightarrow{E} \Sigma'iq$. Hence we obtain the result.

Assertion (2) is a direct consequence of assertion (1). \square

Remark 6.7. Let $\mathbf{E}_1 = (\mathcal{E}_1, w_1) \xrightarrow{i} \mathbf{E}_2 = (\mathcal{E}_2, w_2) \xrightarrow{p} \mathbf{E}_3 = (\mathcal{E}_3, w_3)$ be a right (resp. left) quasi-exact sequence of strict relative exact categories with a right (resp. left) quasi-splitting (j, q, A, B, C, D) . Then

(1) The sequence $\mathbf{Ch}_b(i) \mathbf{Ch}_b(q) \xrightarrow{\mathbf{Ch}_b(A)} \text{id}_{\mathbf{Ch}_b(\mathcal{E}_2)} \xrightarrow{\mathbf{Ch}_b(C)} \mathbf{Ch}_b(j) \mathbf{Ch}_b(p)$ is an admissible exact sequence of exact endofunctors on $\mathbf{Ch}_b(\mathcal{E}_2)$ by 2.11 (3).

(2) Let a pair $(\tilde{\omega}_{\mathbf{Ch}_b(\mathcal{E}_2)}, \partial) : \mathbf{Ch}_b(\mathcal{E}_2) \rightarrow \mathcal{T}(\mathbf{Ch}_b(\mathcal{E}_2)) (= \mathcal{D}_b(\mathbf{E}_2))$ be a widely exact functor in 3.15 (3). We put $E := \partial_{\mathbf{Ch}_b(A), \mathbf{Ch}_b(C)}$. Then the sequence of triangulated categories

$\mathcal{D}_b(\mathbf{E}_1) \xrightarrow{\mathcal{D}_b(i)} \mathcal{D}_b(\mathbf{E}_2) \xrightarrow{\mathcal{D}_b(p)} \mathcal{D}_b(\mathbf{E}_3)$ is a right (resp. left) quasi-split exact sequence with a right (resp. left) quasi-splitting $(\mathcal{D}_b(j), \mathcal{D}_b(q), \mathcal{D}_b(A), \mathcal{D}_b(B), \mathcal{D}_b(C), \mathcal{D}_b(D), E)$.

(3) In particular, a right (resp. left) quasi-split exact sequence of strict relative exact categories is a derived right (resp. left) quasi-split exact sequence.

Example 6.8 (Quasi-split exact sequences). (1) We enumerate examples of quasi-split exact sequences from [Sch11, A.2.8].

(i) Let $p : \mathcal{T} \rightarrow \mathcal{T}'$ be a triangle functor which admits a right adjoint functor $j : \mathcal{T}' \rightarrow \mathcal{T}$ with adjunction morphisms $D : pj \rightarrow \text{id}_{\mathcal{T}'}$ and $C : \text{id}_{\mathcal{T}} \rightarrow jp$ such that D is a natural equivalence

and $i : \ker p \rightarrow \mathcal{T}$ a natural inclusion functor. Then the sequence $\ker p \xrightarrow{i} \mathcal{T} \xrightarrow{p} \mathcal{T}'$ is a right quasi-split exact sequence.

(ii) Let \mathcal{T} be a triangulated category and \mathcal{T}_0 and \mathcal{T}_1 triangulated subcategories of \mathcal{T} such that the following conditions hold.

(a) $\mathcal{T} = \mathcal{T}_0 \vee \mathcal{T}_1$.

(b) $\text{Hom}_{\mathcal{T}}(x, y) = 0$ for any objects x in \mathcal{T}_0 and y in \mathcal{T}_1 .

We write $i_j : \mathcal{T}_j \rightarrow \mathcal{T}$ and $\pi_j : \mathcal{T} \rightarrow \mathcal{T} / \mathcal{T}_j$ for the natural inclusion functor and the natural quotient functor for $j = 0, 1$. Then the composition $x : \mathcal{T}_1 \xrightarrow{i_1} \mathcal{T} \xrightarrow{\pi_0} \mathcal{T} / \mathcal{T}_0$ is an equivalence of triangulated categories and the sequence $\mathcal{T}_0 \xrightarrow{i_0} \mathcal{T} \xrightarrow{\pi_1} \mathcal{T}_1$ is a right quasi-split exact sequence.

(2) Now we illustrate the typical example of Bousfield-Neeman localization of triangulated categories. We assume that readers are familiar with Voevodsky's motive category, for example [MVW06, §14]. Let k be a perfect field and $\mathcal{D}^-(\text{Sh}_{\text{Nis}}(\text{SmCor}(k)))$ the derived category of Nisnevich sheaves with transfers over k , $\text{DM}^{\text{eff}}(k)$ the full subcategory of \mathbb{A}^1 -local objects in $\mathcal{D}^-(\text{Sh}_{\text{Nis}}(\text{SmCor}(k)))$ and $j : \text{DM}^{\text{eff}}(k) \hookrightarrow \mathcal{D}^-(\text{Sh}_{\text{Nis}}(\text{SmCor}(k)))$ the inclusion functor. Then j has a left adjoint functor $C_* : \mathcal{D}^-(\text{Sh}_{\text{Nis}}(\text{SmCor}(k))) \rightarrow \text{DM}^{\text{eff}}(k)$ such that the adjunction morphism $C_* j \rightarrow \text{id}_{\text{DM}^{\text{eff}}(k)}$ is a natural equivalence. (See Ibid). Therefore the sequence $\ker C_* \rightarrow \mathcal{D}^-(\text{Sh}_{\text{Nis}}(\text{SmCor}(k))) \xrightarrow{C_*} \text{DM}^{\text{eff}}(k)$ is a right quasi-split exact sequence by (1) (i).

Definition 6.9 (Extension closed subcategory). Let $\mathcal{R} \hookrightarrow \mathcal{R}'$ be full subcategories of RelEx such that \mathcal{R}' contains the trivial relative exact category 0. We say that \mathcal{R} is **closed under extensions** in \mathcal{R}' if it contains the trivial relative exact category 0 and if for any right quasi-split exact sequence of relative exact categories $\mathbf{E}_1 \rightarrow \mathbf{E}_2 \rightarrow \mathbf{E}_3$ in \mathcal{R}' , if \mathbf{E}_1 and \mathbf{E}_3 are in \mathcal{R} , \mathbf{E}_2 is also in \mathcal{R} .

Proposition 6.10. $\text{RelEx}_{\text{consist}}$ is closed under extensions in $\text{RelEx}_{\text{strict}}$.

Proof. Let $\mathbf{E}_1 \xrightarrow{i} \mathbf{E}_2 \xrightarrow{p} \mathbf{E}_3$ be a right (resp. left) quasi-split exact sequence of strict relative exact category and assume that \mathbf{E}_1 and \mathbf{E}_3 are consistent. We put $\mathbf{E}_i = (\mathcal{E}_i, w_i)$ for $i = 1, 2$ and 3 and let (j, q, A, B, C, D) be a right quasi-splitting of the sequence (i, p) . For any morphism $a : x \rightarrow y$ in w_2 , we consider the commutative diagram of admissible exact sequences in $\text{Ch}_b(\mathcal{E}_2)$

$$\begin{array}{ccccc} \text{Ch}_b(i)j_{\mathcal{E}_1}qx & \xrightarrow{\quad} & j_{\mathcal{E}_2}x & \twoheadrightarrow & \text{Ch}_b(j)j_{\mathcal{E}_3}px \\ \text{Ch}_b(i)j_{\mathcal{E}_1}(qa) \downarrow & & j_{\mathcal{E}_2}a \downarrow & & \downarrow \text{Ch}_b(j)j_{\mathcal{E}_3}pa \\ \text{Ch}_b(i)j_{\mathcal{E}_1}qy & \xrightarrow{\quad} & j_{\mathcal{E}_2}y & \twoheadrightarrow & \text{Ch}_b(j)j_{\mathcal{E}_3}pa. \end{array}$$

Here both $\text{Ch}_b(i)j_{\mathcal{E}_1}(qa)(= j_{\mathcal{E}_2}iqa)$ and $\text{Ch}_b(j)j_{\mathcal{E}_3}pa(= j_{\mathcal{E}_2}jpa)$ are in qw_2 by assumption. Since qw_2 is closed under extensions by 3.20 (1), $j_{\mathcal{E}_2}a$ is also in qw_2 . Hence we obtain the result. \square

Definition 6.11 (Flag). (1) A **right** (resp. **left**) **flag** of a triangulated category \mathcal{T} is a finite sequence of fully faithful functors

$$\{0\} = \mathcal{T}_0 \xrightarrow{k_0} \mathcal{T}_1 \xrightarrow{k_1} \mathcal{T}_2 \xrightarrow{k_2} \cdots \xrightarrow{k_{n-1}} \mathcal{T}_n = \mathcal{T}$$

such that for any i , the canonical sequence $\mathcal{T}_i \xrightarrow{k_i} \mathcal{T}_{i+1} \rightarrow \mathcal{T}_{i+1} / \mathcal{T}_i$ is a right (resp. left) quasi-split exact sequence.

(2) A **derived right** (resp. **left**) **flag** of a relative exact category \mathbf{E} is a finite sequence of derived fully faithful relative exact functors

$$\{0\} = \mathbf{E}_0 \xrightarrow{k_0} \mathbf{E}_1 \xrightarrow{k_1} \mathbf{E}_2 \xrightarrow{k_2} \dots \xrightarrow{k_{n-1}} \mathbf{E}_n = \mathbf{E}$$

such that the induced sequence of triangulated categories

$$\{0\} = \mathcal{D}_b(\mathbf{E}_0) \xrightarrow{\mathcal{D}_b(k_0)} \mathcal{D}_b(\mathbf{E}_1) \xrightarrow{\mathcal{D}_b(k_1)} \mathcal{D}_b(\mathbf{E}_2) \xrightarrow{\mathcal{D}_b(k_2)} \dots \xrightarrow{\mathcal{D}_b(k_{n-1})} \mathcal{D}_b(\mathbf{E}_n) = \mathcal{D}_b(\mathbf{E})$$

is a right (resp. left) flag of $\mathcal{D}_b(\mathbf{E})$.

Example 6.12. Let A be a commutative noetherian ring with unit and $\mathbf{E} = (\text{Perf}_{\mathbb{P}_A^n}, \text{qis})$ the relative exact category of perfect complexes over \mathbb{P}_A^n . For any integer $0 \leq k \leq n$, we let \mathcal{T}_k and \mathcal{T}'_k be the thick subcategories of $\mathcal{T}(\mathbf{E}) \xrightarrow{\sim} \mathcal{D}_b(\mathbf{E})$ spanned by $\mathcal{O}(l)$ where $-k \leq l \leq 0$ and $\mathcal{O}(-k)$ respectively and \mathcal{E}_k the pull-back of \mathcal{T}_k by the projection functor $\text{Perf}_{\mathbb{P}_A^n} \rightarrow \mathcal{T}(\mathbf{E}) \xrightarrow{\sim} \mathcal{D}_b(\mathbf{E})$ and we put $\mathbf{E}_k = (\mathcal{E}_k, \text{qis})$. Then we can easily check that the subcategories \mathcal{T}_{k-1} , $\mathcal{T}'_k \hookrightarrow \mathcal{T}_k$ satisfy the conditions in 6.8 (1) (ii). (See [Sch11, 3.5.1]). Therefore the sequence of the inclusion functors

$$0 \rightarrow \mathbf{E}_1 \rightarrow \mathbf{E}_2 \rightarrow \dots \rightarrow \mathbf{E}_n = \mathbf{E}$$

is a derived right flag.

7 Additive and localizing theories

In this section, we will axiomize both K^W - and \mathbb{K} -theories for suitable relative exact categories. We will describe the relationship between the Gillet-Waldhausen type formula and the fibration theorem in 7.4. Using this observation and the results in previous sections, we will bring examples of relative exact categories which behave well to K^W - and \mathbb{K} -theories in 7.5 and in 7.6. We will also formulate additive and localizing theories for certain relative exact categories and straighten up their fundamental properties in 7.9 and 7.10 respectively.

Definition 7.1 (Reasonable subcategories). A full subcategory \mathcal{R} of $\mathbf{RelEx}_{\text{strict}}$ is **reasonable** if \mathcal{R} contains the trivial relative exact category 0 and if for any relative exact categories $\mathbf{E} = (\mathcal{E}, w)$ in \mathcal{R} , $(\mathcal{E}, i_{\mathcal{E}})$, $(\mathcal{E}^w, i_{\mathcal{E}^w})$ and $\mathbf{Ch}_b \mathbf{E}$ are also in \mathcal{R} .

Remark 7.2. Let F be a functor from a reasonable subcategory \mathcal{R} of $\mathbf{RelEx}_{\text{strict}}$ to a category \mathcal{X} and \mathcal{E} an exact category such that $(\mathcal{E}, i_{\mathcal{E}})$ is in \mathcal{R} . We sometimes write $F(\mathcal{E})$ for $F(\mathcal{E}, i_{\mathcal{E}})$. We say that a relative exact functor $f : \mathbf{F} \rightarrow \mathbf{G}$ in \mathcal{R} is a **F -equivalence** if $F(f) : F(\mathbf{F}) \rightarrow F(\mathbf{G})$ is an isomorphism in \mathcal{X} .

Example 7.3 (Reasonable subcategories). The categories $\mathbf{RelEx}_{?}^{\#}$ and $\mathbf{WalEx}_{?}^{\#}$ for $\# \in \{+, \text{nothing}\}$ and $? \in \{\text{consist}, \text{solid}\}$ are reasonable subcategories by 3.20 and 4.13.

Lemma-Definition 7.4 (Excellent relative exact categories). Let \mathcal{R} be a reasonable subcategory of $\mathbf{RelEx}_{\text{consist}}$ and F a functor from \mathcal{R} to the stable category of spectra and $\mathbf{E} = (\mathcal{E}, w)$ a consistent relative exact category in \mathcal{R} . We assume the following axiom holds.

(F -weak Gillet-Waldhausen axiom). The canonical functors

$$j_{\mathcal{E}} : (\mathcal{E}, i_{\mathcal{E}}) \rightarrow (\mathbf{Ch}_b(\mathcal{E}), \text{qis}) \text{ and}$$

$$j_{\mathcal{E}^w} : (\mathcal{E}^w, i_{\mathcal{E}^w}) \rightarrow (\mathbf{Ch}_b(\mathcal{E}^w), \text{qis})$$

are F -equivalences.

Then two of the following axioms imply the other axiom.

(F -Gillet-Waldhausen axiom). The canonical functors $j_{\mathcal{E}} : \mathbf{E} \rightarrow \mathbf{Ch}_b(\mathbf{E})$ is a F -equivalence.

(F -weak fibration axiom). The inclusion functor $\mathcal{E}^w \hookrightarrow \mathcal{E}$ and the identity functor of \mathcal{E} induces a fibration sequence of spectra

$$F(\mathbf{Ch}_b(\mathcal{E}^w), \text{qis}) \rightarrow F(\mathbf{Ch}_b(\mathcal{E}), \text{qis}) \rightarrow F(\mathbf{Ch}_b(\mathbf{E})).$$

(F -fibration axiom). The inclusion functor $\mathcal{E}^w \hookrightarrow \mathcal{E}$ and the identity functor of \mathcal{E} induces a fibration sequence of spectra

$$F(\mathcal{E}^w, i_{\mathcal{E}^w}) \rightarrow F(\mathcal{E}, i_{\mathcal{E}}) \rightarrow F(\mathbf{E}).$$

We say that \mathbf{E} is F -**excellent** if \mathbf{E} satisfies four axioms above. We write $F - \mathcal{R}$ for the full subcategory of F -excellent relative exact categories in \mathcal{R} .

Proof. Consider the commutative diagram below.

$$\begin{array}{ccccc} F(\mathcal{E}^w, i_{\mathcal{E}^w}) & \longrightarrow & F(\mathcal{E}, i_{\mathcal{E}}) & \longrightarrow & F(\mathbf{E}) \\ F(j_{\mathcal{E}^w}) \downarrow \wr & & F(j_{\mathcal{E}}) \downarrow \wr & & \downarrow F(j_{\mathbf{E}}) \\ F(\mathbf{Ch}_b(\mathcal{E}^w), \text{qis}) & \longrightarrow & F(\mathbf{Ch}_b(\mathcal{E}), \text{qis}) & \longrightarrow & F(\mathbf{Ch}_b(\mathbf{E})). \end{array}$$

Then if we assume both the horizontal lines above are fibration sequences of spectra, then the map $F(j_{\mathcal{E}}) : F(\mathbf{E}) \rightarrow F(\mathbf{Ch}_b(\mathbf{E}))$ is a homotopy equivalence of spectra. Next if we assume the map $F(j_{\mathcal{E}}) : F(\mathbf{E}) \rightarrow F(\mathbf{Ch}_b(\mathbf{E}))$ is a homotopy equivalence of spectra, then the top line is a fibration sequence of spectra if and only if the bottom line is. \square

Proposition 7.5 (Examples of excellent relative exact categories). (1) *A very strict solid Waldhausen exact category which satisfies the K^W -fibration axiom is K^W -excellent.*
(2) *A very strict consistent relative exact category is \mathbb{K} -excellent.*

Proof. Let $\mathbf{E} = (\mathcal{E}, w)$ be a very strict solid Waldhausen (resp. very strict relative) exact category. Then the sequence

$$(\mathbf{Ch}_b(\mathcal{E}^w), \text{qis}) \rightarrow (\mathbf{Ch}_b(\mathcal{E}), \text{qis}) \rightarrow \mathbf{Ch}_b(\mathbf{E})$$

induced from the inclusion functor $\mathcal{E}^w \hookrightarrow \mathcal{E}$ and the identity functor of \mathcal{E} is an exact sequence (resp. weakly exact sequence) of complicial Waldhausen categories by 4.17. Therefore by [Sch11, 3.2.23] (resp. [Sch11, 3.2.27]), \mathbf{E} satisfies the K^W -weak fibration (resp. \mathbb{K} -weak fibration) axiom. On the other hand, for any exact category (resp. consistent relative exact category), it satisfies K^W -weak Gillet-Waldhausen (resp. \mathbb{K} -Gillet-Waldhausen) axiom by [Cis02] (resp. [Sch11, 3.2.29] and 4.15). Hence we obtain the result by 7.4. \square

Example 7.6. (1) Let \mathcal{E} be an exact category. Since $\mathcal{E}^{i_{\mathcal{E}}}$ is trivial, the pair $(\mathcal{E}, i_{\mathcal{E}})$ is a very strict solid Waldhausen exact category which satisfies the K^W -fibration axiom. In particular, it is K^W -excellent and \mathbb{K} -excellent by 7.5.

(2) In [Sch06, Theorem 11], Schlichting showed that a Waldhausen category which satisfies the extensional, the saturated and the factorization axioms, also satisfies the K^W -fibration axiom. In particular, a bicomplial pair satisfies the K^W -fibration axiom by 3.8. Hence a bicomplial pair is K^W -excellent and \mathbb{K} -excellent by 3.9, 4.13 and 7.5.

7.7 (Proof of Theorem 0.3). Let $\mathbf{E} = (\mathcal{E}, w)$ be a consistent relative exact category. Assertion (1) follows from [Sch06, Theorem 8].

(2) If w is the class of all isomorphisms in \mathcal{E} , the definition of $\mathbb{K}(\mathbf{E})$ is compatible with Definition 8 in [Sch06]. If \mathbf{E} is a complicial Waldhausen category, the result follows from 4.15 (2) and [Sch06, Proposition 3].

(3) We have a homotopy equivalence of spectra $\mathbb{K}_n(\mathbf{E}) = \mathbb{K}_n^S(\mathbf{Ch}_b(\mathbf{E})) \xrightarrow{\sim} K_n^W(\mathbf{Ch}_b(\mathbf{E}))$ for any positive integer n by [Sch06, Theorem 8] again. If \mathbf{E} is a very strict solid Waldhausen exact category and satisfies the K^W -fibration axiom, then $j_{\mathcal{E}}$ induces a homotopy equivalence of spectra between $K_n^W(\mathbf{E})$ and $K_n^W(\mathbf{Ch}_b(\mathbf{E}))$ by 7.5. \square

Recall the definition of $E(\mathcal{E})$ and the exact functors s, m, q from $E(\mathcal{E})$ to \mathcal{E} from Conventions (7) (xiv).

Lemma-Definition 7.8 (Additive and localizing theories). (1) (cf. [Wal85, 1.3.2], [GSVW92, 3.1]). Let F be a functor from a full subcategory \mathcal{R} of \mathbf{RelEx} which closed under extensions to an additive category \mathcal{B} . Each of the following assertions implies all the three others.

(i) For any right quasi-split exact sequence $\mathbf{E} \xrightarrow{i} \mathbf{F} \xrightarrow{p} \mathbf{G}$ in \mathcal{R} with a right quasi-splitting (j, q) , the morphism

$$\begin{pmatrix} F(q) \\ F(p) \end{pmatrix} : F(\mathbf{F}) \rightarrow F(\mathbf{E}) \oplus F(\mathbf{G})$$

is an isomorphism.

(ii) For any relative exact category \mathbf{E} in \mathcal{R} , the following projection is an isomorphism

$$\begin{pmatrix} F(s) \\ F(q) \end{pmatrix} : F(E(\mathbf{E})) \rightarrow F(\mathbf{E}) \oplus F(\mathbf{E}).$$

(iii) For any relative exact category \mathbf{E} in \mathcal{R} , we have the equality $F(m) = F(s) + F(q)$ for morphisms $F(E(\mathbf{E})) \rightarrow F(\mathbf{E})$.

(iv) For any admissible exact sequence $f \xrightarrow{A} g \xrightarrow{B} h$ of relative exact functors between relative exact categories $\mathbf{E} \rightarrow \mathbf{F} = (\mathcal{F}, v)$ in \mathcal{R} , we have the equality $F(g) = F(f) + F(h)$.

We say that F is an **additive theory** if F satisfies the assertions above.

(2) A functor F from a reasonable subcategory \mathcal{R} of $\mathbf{RelEx}_{\text{strict}}$ to the stable category of spectra is a **localizing** (resp. **strictly localizing**) **theory** if F sends a weakly exact (resp. exact) sequence to a fibration sequence of spectra. \square

Proposition 7.9. Let \mathcal{R} be a full 2-subcategory of \mathbf{RelEx}^+ which closed under extensions and F is an additive theory from \mathcal{R} to an additive category \mathcal{B} . Then

(1) The canonical map $F(0) \rightarrow 0$ is an isomorphism.

(2) Assume that F is categorical homotopy invariant. Then for any bicomplial pair $\mathbf{C} = (\mathcal{C}, v)$ with the suspension functor $T : \mathcal{C} \rightarrow \mathcal{C}$ in \mathcal{R} , we have the equality $F(T) = -\text{id}_{F(\mathcal{C})}$.

Proof. (1) Consider the right quasi-split sequence $0 \rightarrow 0 \rightarrow 0$. Then we obtain the isomorphism $F(0) \xrightarrow{\sim} F(0) \oplus F(0)$ by additivity. Hence we obtain the result.

(2) By applying 7.8 (1) (iv) to the admissible exact sequence $\text{id}_{\mathcal{C}} \rightarrow \mathcal{C} \rightarrow T$ on \mathcal{C} , we get the equality $\text{id}_{F(\mathcal{C})} + F(T) = F(\mathcal{C})$. On the other hand, since the natural transformation $0 \rightarrow \mathcal{C}$ is a relative natural equivalence, $F(\mathcal{C}) = 0$ by homotopy invariance of F . \square

Proposition 7.10. Let F be a localizing (resp. strictly localizing) theory on a reasonable subcategory \mathcal{R} of $\mathbf{RelEx}_{\text{strict}}$. Then

- (1) The canonical map $F(0) \rightarrow 0$ is a homotopy equivalence of spectra.
- (2) **(Approximation theorem).** Let $f : \mathbf{E} \rightarrow \mathbf{F}$ be a relative exact functor in \mathcal{R} . If f is a weakly derived equivalence (resp. derived equivalence), then f is a F -equivalence.
- (3) **(Additivity theorem).** Assume that moreover F is a categorical homotopy invariant functor and \mathcal{R} is closed under extensions. Then F is an additivity theory.

Proof. (1) Consider the following commutative diagram of fibration sequences of spectra

$$\begin{array}{ccccc} F(0) & \longrightarrow & F(0) & \longrightarrow & F(0) \\ \parallel & & \parallel & & \downarrow \\ F(0) & \longrightarrow & F(0) & \longrightarrow & 0. \end{array}$$

Then it turns out that $F(0) \rightarrow 0$ is a homotopy equivalent of spectra.

(2) We just apply the assumption of F to a weakly exact (resp. exact) sequence $0 \rightarrow \mathbf{E} \xrightarrow{f} \mathbf{F}$.

(3) let $\mathbf{E} \xrightarrow{i} \mathbf{F} \xrightarrow{p} \mathbf{G}$ be a right quasi-split exact sequence with a right quasi-splitting (q, j) . By 6.6 and 6.7, the sequence (i, p) is exact. Consider the following commutative diagram of fibration sequences of spectra

$$\begin{array}{ccccc} F(\mathbf{E}) & \xrightarrow{F(i)} & F(\mathbf{F}) & \xrightarrow{F(p)} & F(\mathbf{G}) \\ \parallel & & \star & & \parallel \\ & & \downarrow \begin{pmatrix} F(q) \\ F(p) \end{pmatrix} & & \\ F(\mathbf{E}) & \xrightarrow{\begin{pmatrix} \text{id}_{F(\mathbf{E})} \\ 0 \end{pmatrix}} & F(\mathbf{E}) \oplus F(\mathbf{G}) & \xrightarrow{\begin{pmatrix} 0 & \text{id}_{F(\mathbf{G})} \end{pmatrix}} & F(\mathbf{G}). \end{array}$$

Here we use homotopy invariance of F to prove the commutativity of \star . Then we learn that the map $\begin{pmatrix} F(q) \\ F(p) \end{pmatrix} : F(\mathbf{F}) \rightarrow F(\mathbf{E}) \oplus F(\mathbf{G})$ is a homotopy equivalence of spectra. \square

Corollary 7.11. Let F be a functor from a reasonable subcategory \mathcal{R} of $\mathbf{RelEx}_{\text{consist}}$ to the stable category of spectra. Assume that the following three conditions hold.

- (1) For any essentially small exact category \mathcal{E} , $(\mathcal{E}, i_{\mathcal{E}})$ is in $F - \mathcal{R}$.
- (2) $\mathbf{Ch}_b(F - \mathcal{R}) \subset F - \mathcal{R}$.
- (3) The functor $F \mathbf{Ch}_b$ on $F - \mathcal{R}$ is a localizing (resp. strictly localizing) theory.
- Then F is a localizing (resp. strictly localizing) theory on $F - \mathcal{R}$.

Proof. For any weakly exact (resp. exact) sequence $\mathbf{E} = (\mathcal{E}, u) \xrightarrow{i} \mathbf{F} = (\mathcal{F}, v) \xrightarrow{j} \mathbf{G} = (\mathcal{G}, w)$ in $F - \mathcal{R}$, we have the commutative diagram in the stable category of spectra below

$$\begin{array}{ccccc} F(\mathbf{E}) & \xrightarrow{F(i)} & F(\mathbf{F}) & \xrightarrow{F(j)} & F(\mathbf{G}) \\ j_{\mathcal{E}} \downarrow & & j_{\mathcal{F}} \downarrow & & j_{\mathcal{G}} \downarrow \\ F(\mathbf{Ch}_b(\mathbf{E})) & \xrightarrow{F(\mathbf{Ch}_b(i))} & F(\mathbf{Ch}_b(\mathbf{F})) & \xrightarrow{F(\mathbf{Ch}_b(j))} & F(\mathbf{Ch}_b(\mathbf{G})). \end{array}$$

Here the bottom line is a fibration sequence by assumption and all vertical lines are homotopy equivalence by F -Gillet-Waldhausen axiom. Therefore we get the desired fibration sequence

$$F(\mathbf{E}) \xrightarrow{F(i)} F(\mathbf{F}) \xrightarrow{F(j)} F(\mathbf{G}).$$

□

Example 7.12. Exact categories and bicomplial pairs are K^W -excellent by 7.6. Moreover $K^W \mathbf{Ch}_b$ is a strongly localizing theory on $K^W - \mathbf{RelEx}_{\text{consist}}$ by 4.17 (1) and [Sch11, 3.2.23]. Hence the functor K^W is a strictly localizing theory on $K^W - \mathbf{RelEx}_{\text{consist}}$ by 7.11.

8 Multi semi-direct products

In this section, we elaborate the theory of multi semi-direct products of exact categories as a continuation of [Moc13]. In the last of this section, we will get 8.19 which is an abstraction of Corollary 5.14 in [Moc13].

8.1. For a set S , an S -**cube** in a category \mathcal{C} is a contravariant functor from $\mathcal{P}(S)$ to \mathcal{C} . We denote the category of S -cubes in a category \mathcal{C} by $\mathbf{Cub}^S \mathcal{C}$ where morphisms between cubes are just natural transformations. Let x be an S -cube in \mathcal{C} . For any $T \in \mathcal{P}(S)$, we denote $x(T)$ by x_T and call it a **vertex of x (at T)**. For $k \in T$, we also write $d_T^{x,k}$ or shortly d_T^k for $x(T \setminus \{k\} \hookrightarrow T)$ and call it a **$(k-)$ boundary morphism of x (at T)**. An S -cube x is **monic** if for any pair of subsets $U \subset T$ in S , $x(U \subset V)$ is a monomorphism.

In the rest of this section, we assume that S is a finite set.

8.2 (Admissible cubes). Fix an S -cube x in an abelian category \mathcal{A} . For any element k in S , the **k -direction 0-th homology** of x is an $S \setminus \{k\}$ -cube $H_0^k(x)$ in \mathcal{A} and defined by $H_0^k(x)_T := \text{Coker } d_{T \cup \{k\}}^k$. For any $T \in \mathcal{P}(S)$ and $k \in S \setminus T$, we denote the canonical projection morphism $x_T \rightarrow H_0^k(x)_T$ by $\pi_T^{k,x}$ or simply π_T^k . When $\#S = 1$, we say that x is **admissible** if x is monic, namely if its unique boundary morphism is a monomorphism. For $\#S > 1$, we define the notion of an admissible cube inductively by saying that x is **admissible** if x is monic and if for every k in S , $H_0^k(x)$ is admissible. If x is admissible, then for any distinct elements i_1, \dots, i_k in S and for any automorphism σ of the set $\{i_1, \dots, i_k\}$, the identity morphism on x induces an isomorphism:

$$H_0^{i_1}(H_0^{i_2}(\dots(H_0^{i_k}(x))\dots)) \xrightarrow{\sim} H_0^{i_{\sigma(1)}}(H_0^{i_{\sigma(2)}}(\dots(H_0^{i_{\sigma(k)}}(x))\dots))$$

where σ is a bijection on S . (cf. [Moc13, 3.11]). For an admissible S -cube x and a subset $T = \{i_1, \dots, i_k\} \subset S$, we put $H_0^T(x) := H_0^{i_1}(H_0^{i_2}(\dots(H_0^{i_k}(x))\dots))$ and $H_0^\emptyset(x) = x$. Notice that $H_0^T(x)$ is an $S \setminus T$ -cube for any $T \in \mathcal{P}(S)$. Then we have the isomorphisms

$$H_p(\text{Tot}(x)) \xrightarrow{\sim} \begin{cases} H_0^S(x) & \text{for } p = 0 \\ 0 & \text{otherwise} \end{cases} . \quad (3)$$

See [Moc13, 3.13].

In the rest of this section, let U and V be a pair of disjoint subsets of S .

Definition 8.3 (Multi semi-direct products). Let $\mathfrak{F} = \{\mathcal{F}_T\}_{T \in \mathcal{P}(S)}$ be a family of full subcategories of \mathcal{A} .

(1) We put $\mathfrak{F}|_U^V := \{\mathcal{F}_{V \sqcup T}\}_{T \in \mathcal{P}(U)}$ and call it the **restriction of \mathfrak{F} (to U along V)**.

(2) Then we define $\ltimes \mathfrak{F} = \ltimes_{T \in \mathcal{P}(S)} \mathcal{F}_T$ the **multi semi-direct products of the family \mathfrak{F}** as

follows. $\ltimes \mathfrak{F}$ is the full subcategory of $\mathbf{Cub}^S(\mathcal{A})$ consisting of those cube x such that x is admissible and each vertex of $H_0^T(x)$ is in \mathcal{F}_T for any $T \in \mathcal{P}(S)$.

(3) If S is a singleton (namely $\#S = 1$), then we write $\mathcal{F}_S \ltimes \mathcal{F}_\emptyset$ for $\ltimes \mathfrak{F}$.

Remark 8.4. For any element u in U , we have the equality $\ltimes \mathfrak{F}|_U^V = (\ltimes \mathfrak{F}|_{U \setminus \{u\}}^{V \sqcup \{u\}}) \ltimes (\ltimes \mathfrak{F}|_{U \setminus \{u\}}^V)$. (See [Moc13, 3.19]).

Definition 8.5 (Exact family). Let $\mathfrak{F} = \{\mathcal{F}_T\}_{T \in \mathcal{P}(S)}$ be a family of strict exact subcategories of an abelian category \mathcal{A} . We say that \mathfrak{F} is an **exact family** (of \mathcal{A}) if for any disjoint pair of subsets P and Q of S , $\ltimes \mathfrak{F}|_P^Q$ is a strict exact subcategory of $\mathbf{Cub}^P \mathcal{A}$.

Lemma 8.6. (cf. [Moc13, 3.20]). Let $\mathfrak{F} = \{\mathcal{F}_T\}_{T \in \mathcal{P}(S)}$ be a family of full subcategories of \mathcal{A} . If \mathcal{F}_T is closed under either extensions or taking sub- and quotient objects and direct sums in \mathcal{A} , then \mathfrak{F} is an exact family. \square

In the rest of this section, let $\mathfrak{F} = \{\mathcal{F}_T\}_{T \in \mathcal{P}(S)}$ be an exact family of \mathcal{A} .

Lemma-Definition 8.7. For any non-empty subset W of U and $j = 0$ or 1 , we will define

$$\text{ext}_{U \setminus W, W}^V : \ltimes \mathfrak{F}|_{U \setminus W}^V \rightarrow \ltimes \mathfrak{F}|_U^V,$$

$$\text{res}_{U, W}^{V, j} : \ltimes \mathfrak{F}|_U^V \rightarrow \ltimes \mathfrak{F}|_{U \setminus W}^V \quad \text{and}$$

$$H_U^{V, W} : \ltimes \mathfrak{F}|_U^V \rightarrow \ltimes \mathfrak{F}|_{U \setminus W}^{V \sqcup W}$$

to be exact functors by induction on the cardinality of W and call them the **extension functor**, the **restriction functor** and the **homology functor**. First assume that W is a singleton $W = \{w\}$. Then we have the equality $\ltimes \mathfrak{F}|_U^V = (\ltimes \mathfrak{F}|_{U \setminus W}^{V \sqcup W}) \ltimes (\ltimes \mathfrak{F}|_{U \setminus W}^V)$ by 8.4. Regarding $\ltimes \mathfrak{F}|_U^V$ as a subcategory of one dimensional cubes in $\ltimes \mathfrak{F}|_{U \setminus W}^V$, we define the three exact functors $\text{ext}_{U \setminus W, W}^V$, $\text{res}_{U, W}^{V, j}$ and $H_U^{V, W}$ by sending an object x in $\ltimes \mathfrak{F}|_{U \setminus W}^V$ and $[x_1 \xrightarrow{d^x} x_0]$ in $\ltimes \mathfrak{F}|_U^V$ to $[x \xrightarrow{\text{id}_x} x]$, x_j and $\text{Coker } d^x$ respectively.

Next for any non-trivial disjoint decomposition of $W = W_1 \sqcup W_2$, we put

$$\text{ext}_{U \setminus W, W}^V := \text{ext}_{U \setminus W_2, W_2}^V \text{ext}_{U \setminus W, W_1}^V,$$

$$\text{res}_{U, W}^{V, j} := \text{res}_{U \setminus W_1, W_2}^{V, j} \text{res}_{U, W_1}^{V, j} \quad \text{and}$$

$$H_U^{V, W} := H_{U \setminus W_1}^{V \sqcup W_1, W_2} H_U^{V, W_1}.$$

Then the definitions of $\text{ext}_{U \setminus W, W}^V$, $\text{res}_{U, W}^{V, j}$ and $H_U^{V, W}$ do not depend upon a choice of disjoint decomposition of W up to canonical isomorphisms and they are exact functors. \square

we can easily prove the following lemma.

Lemma 8.8. (1) For any non-empty subset W of U , we have the equality

$$\text{res}_{U, W}^{V, j} \text{ext}_{U \setminus W}^{V, W} = \text{id}_{\ltimes \mathfrak{F}|_{U \setminus W}^V}$$

for $j = 0, 1$.

(2) For any pair of disjoint non-empty subsets W_1 and W_2 of U , we have the equality

$$H_U^{V, W_1} \text{ext}_{U \setminus W_2}^{V, W_2} = \text{ext}_{U \setminus W_1}^{V \sqcup W_1, W_2} H_U^{V, W_1}.$$

□

In the rest of this section, let w be a class of morphisms in \mathcal{F}_S and assume that for any zero objects 0 and $0'$ in \mathcal{F}_S , the canonical morphism $0 \rightarrow 0'$ is in w .

Definition 8.9. We will define the class of **total weak equivalences** $tw(\times \mathfrak{F}|_U^V)$ or simply tw in $\times \mathfrak{F}|_U^V$. First assume that $S = U \sqcup V$. Then tw is defined by pull-back of w by the exact functor $H_U^{V, U} : \times \mathfrak{F}|_U^V \rightarrow \mathcal{F}_S$.

Next we assume that $U \sqcup V \neq S$. Then tw is defined by pull-back of $tw(\times \mathfrak{F}|_{S \setminus V}^V)$ by the exact functor $\text{ext}_{U, S \setminus (U \sqcup V)}^V : \times \mathfrak{F}|_U^V \rightarrow \times \mathfrak{F}|_{S \setminus V}^V$.

Remark 8.10. (1) (cf. [Moc13, 3.19]). For any element u of U , we have the equality

$$(\times \mathfrak{F}|_U^V)^{tw} = (\times \mathfrak{F}|_{U \setminus \{u\}}^{V \sqcup \{u\}})^{tw} \times (\times \mathfrak{F}|_{U \setminus \{u\}}^V).$$

(2) For any non-empty subset W of U and $j = 0$ or 1 , the exact functors $\text{ext}_{U \setminus W, W}^V$, $\text{res}_{U, W}^{V, j}$ and $H_U^{V, W}$ preserve tw .

We can easily prove the following lemma.

Lemma 8.11. *if w satisfies the extensional (resp. gluing, cogluing) axiom, then tw in $\times \mathfrak{F}|_U^V$ also satisfies the axiom.* □

Definition 8.12. We say that w is **compatible with** \mathfrak{F} if for any disjoint pair of subsets P and Q of S , $(\times \mathfrak{F}|_P^Q, tw)$ is a strict relative exact category.

Corollary 8.13. *If either (\mathcal{F}_S, w) satisfies the extensional axiom or (\mathcal{F}_S, w) is a Waldhausen exact category, then w is compatible with \mathfrak{F} .*

Proof. If either (\mathcal{F}_S, w) satisfies the extensional axiom or (\mathcal{F}_S, w) is a Waldhausen exact category, then $(\times \mathfrak{F}|_P^Q, tw)$ is also for any pair of disjoint subsets P and Q of S by 8.11. Hence $(\times \mathfrak{F}|_P^Q, tw)$ is strict by 2.4. □

Definition 8.14 (Adroit system). (cf. [Moc13, 2.20]). An **adroit** (resp. a **strongly adroit**) **system** in an abelian category \mathcal{A} is a system $\mathcal{X} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{F})$ consisting of strict exact subcategories $\mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \hookleftarrow \mathcal{F}$ in \mathcal{A} and they satisfies the following axioms **(Adr 1)**, **(Adr 2)**, **(Adr 3)** and **(Adr 4)** (resp. **(Adr 1)**, **(Adr 2)**, **(Adr 3)** and **(Adr 5)**).

(Adr 1) $\mathcal{F} \times \mathcal{E}_1$ and $\mathcal{F} \times \mathcal{E}_2$ are strict exact subcategories of $\text{Ch}_b(\mathcal{A})$.

(Adr 2) \mathcal{E}_1 is closed under extensions in \mathcal{E}_2 .

(Adr 3) Let $x \rightarrowtail y \twoheadrightarrow z$ be an admissible short exact sequence in \mathcal{A} . Assume that y is isomorphic to an object in \mathcal{E}_1 and z is isomorphic to an object in \mathcal{E}_1 or \mathcal{F} . Then x is isomorphic to an object in \mathcal{E}_1 .

(Adr 4) For any object z in \mathcal{E}_2 , there exists an object y in \mathcal{E}_1 and an admissible epimorphism $y \twoheadrightarrow z$.

(Adr 5) For any non-negative integer m and an object z in $\mathcal{E}_2^{[m]}$, there exists an object y in $\mathcal{E}_1^{[m]}$ and an admissible epimorphism $y \twoheadrightarrow z$.

Remark 8.15. Let $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{F})$ be an adroit (resp. a strongly adroit) system of \mathcal{A} and \mathcal{H} a strict exact subcategory of \mathcal{F} . Then a triple $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{H})$ is an adroit (resp. a strongly adroit) system.

Next theorem is a variation of Theorem 2.21 in [Moc13].

Theorem 8.16. Let $\mathcal{X} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{F})$ be an adroit system and v a class of morphisms in \mathcal{F} such that either v satisfies the extension axiom or (\mathcal{F}, v) is a Waldhausen exact category. Then

(1) The canonical inclusion functors

$$\mathcal{F} \ltimes \mathcal{E}_1 \hookrightarrow \mathcal{F} \ltimes \mathcal{E}_2 \text{ and } \mathcal{F}^v \ltimes \mathcal{E}_1 \hookrightarrow \mathcal{F}^v \ltimes \mathcal{E}_2$$

satisfy the resolution conditions. In particular, the inclusion functor

$$(\mathcal{F} \ltimes \mathcal{E}_1, tv) \rightarrow (\mathcal{F} \ltimes \mathcal{E}_2, tv)$$

is a derived equivalence.

(2) Assume that (\mathcal{F}, v) is a consistent relative exact category. Then the relative exact functor $H_0 : (\mathcal{F} \ltimes \mathcal{E}_1, tv) \rightarrow (\mathcal{F}, v)$ is a derived equivalence.

(3) The exact functors $\mathcal{E}_1 \rightarrow \mathcal{F} \ltimes \mathcal{E}_1, x \mapsto [x \xrightarrow{\text{id}_x} x]$ and $H_0 : \mathcal{F} \ltimes \mathcal{E}_1 \rightarrow \mathcal{F}$ yield a right quasi-split exact sequence $(\mathcal{E}_1, i_{\mathcal{E}_1}) \rightarrow (\mathcal{F} \ltimes \mathcal{E}_1, i_{\mathcal{F} \ltimes \mathcal{E}_1}) \rightarrow (\mathcal{F}, i_{\mathcal{F}})$.

Proof. (1) In [Moc13, 2.21], we prove that if \mathcal{X} is an adroit system, then the inclusion functor $\mathcal{F} \ltimes \mathcal{E}_1 \hookrightarrow \mathcal{F} \ltimes \mathcal{E}_2$ satisfies the resolution conditions. Notice that if $\mathcal{X} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{F})$ is an adroit system, then $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{F}^w)$ is also an adroit system by 2.4 and 8.15. Therefore we learn that the inclusion functor $\mathcal{F}^w \ltimes \mathcal{E}_1 \hookrightarrow \mathcal{F}^w \ltimes \mathcal{E}_2$ also satisfies the resolution conditions. The last assertion follows from the derived resolution theorem in 5.3.

(2) First we prove that the exact functor $H_0 : (\mathcal{F} \ltimes \mathcal{E}_2, tv) \rightarrow (\mathcal{F}, v)$ is a categorical homotopy equivalence. This functor has a section $s : \mathcal{F} \rightarrow \mathcal{F} \ltimes \mathcal{E}_2, x \mapsto [0 \rightarrow x]$. Moreover for any $x = [x_1 \xrightarrow{d^x} x_0]$ in $\mathcal{F} \ltimes \mathcal{E}_2$, the canonical quotient morphism $x_1 \rightarrow H_0(x)$ induces the relative natural equivalence $\text{id}_{\mathcal{F} \ltimes \mathcal{E}_2} \rightarrow s H_0$. Hence $H_0 : (\mathcal{F} \ltimes \mathcal{E}_2, tv) \rightarrow (\mathcal{F}, v)$ is a categorical homotopy equivalence and it is a derived equivalence by 3.21 (2). It turns out that the composition $(\mathcal{F} \ltimes \mathcal{E}_1, tv) \rightarrow (\mathcal{F} \ltimes \mathcal{E}_2, tv) \xrightarrow{H_0} (\mathcal{F}, v)$ is also a derived equivalence by (1).

(3) Consider the commutative diagram below:

$$\begin{array}{ccccc} (\mathcal{E}_1, i_{\mathcal{E}_1}) & \longrightarrow & (\mathcal{F} \ltimes \mathcal{E}_1, i_{\mathcal{F} \ltimes \mathcal{E}_1}) & \longrightarrow & (\mathcal{F}, i_{\mathcal{F}}) \\ \downarrow & & \downarrow & & \parallel \\ (\mathcal{E}_1, i_{\mathcal{E}_2}) & \longrightarrow & (\mathcal{F} \ltimes \mathcal{E}_1, i_{\mathcal{F} \ltimes \mathcal{E}_2}) & \longrightarrow & (\mathcal{F}, i_{\mathcal{F}}). \end{array}$$

Here the vertical lines are induced from the inclusion functors and they are derived equivalences by (1). The bottom horizontal line is a right quasi-split exact sequence by [Moc13, 2.19]. Hence we obtain the result. \square

Lemma 8.17. Let $\mathcal{F} \hookrightarrow \mathcal{E}$ be strict exact subcategories of \mathcal{A} and v a class of morphisms in \mathcal{F} . If (\mathcal{F}, v) is solid, then $(\mathcal{F} \ltimes \mathcal{E}, tv)$ is also solid.

Proof. We consider an object x in $\mathcal{F} \ltimes \mathcal{E}$ to be a complex $[x_1 \xrightarrow{d^x} x_0]$ in $\mathbf{Ch}_b(\mathcal{E})$. For any morphism $f : x \rightarrow y$ in $tv(\mathcal{F} \ltimes \mathcal{E})$, $H_0(f) : H_0(x) \rightarrow H_0(y)$ is in v . Therefore $\text{Cone } H_0(f)$ is in $\text{Acy}_b^{qv}(\mathcal{F})$ by assumption. Notice that there exists the admissible exact sequence

$$\text{Cone } \text{Cone}(f_1, f_1) \twoheadrightarrow \text{Cone } f \twoheadrightarrow \mathbf{Ch}_b(s)(\text{Cone } H_0(f))$$

in $\mathbf{Ch}_b(\mathcal{F} \ltimes \mathcal{E})$ where $s : \mathcal{F} \rightarrow \mathcal{F} \ltimes \mathcal{E}$ is the exact functor defined by sending an object x in \mathcal{F} to an object $[x \xrightarrow{\text{id}_x} x]$ in $\mathcal{F} \ltimes \mathcal{E}$. Namely we consider the following commutative diagram.

$$\begin{array}{ccccccc}
 & & x_1 & \xrightarrow{\text{id}_{x_1}} & x_1 & \xrightarrow{\quad} & 0 \\
 & \swarrow \text{id}_{x_1} & \downarrow d^x & \swarrow d^x & \downarrow & \swarrow & \downarrow \\
 x_1 & \xrightarrow{\quad} & x_0 & \xrightarrow{\quad} & H_0 x & \xrightarrow{\quad} & 0 \\
 \downarrow f_1 & & \downarrow & & \downarrow & & \downarrow \\
 & \swarrow \text{id}_{y_1} & y_1 & \xrightarrow{f_0} & y_1 & \xrightarrow{\quad} & 0 \\
 & \downarrow & \downarrow d^y & \swarrow d^y & \downarrow & \swarrow & \downarrow \\
 y_1 & \xrightarrow{\quad} & y_0 & \xrightarrow{\quad} & H_0 y & \xrightarrow{\quad} & 0
 \end{array}$$

Then $\text{Cone} \text{Cone}(f_1, f_1)$ and $\mathbf{Ch}_b(s)(\text{Cone } H_0(f))$ are in $\text{Acy}_b^{qtv}(\mathcal{F} \ltimes \mathcal{E})$. Since $\text{Acy}_b^{qtv}(\mathcal{F} \ltimes \mathcal{E})$ is closed under extensions, $\text{Cone } f$ is also in $\text{Acy}_b^{qtv}(\mathcal{F} \ltimes \mathcal{E})$. This means that $(\mathcal{F} \ltimes \mathcal{E}, tv)$ satisfies the solid axiom. \square

Corollary 8.18. *Let $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{F})$ is an adroit system of \mathcal{A} and v a class of morphisms in \mathcal{F} . If (\mathcal{F}, v) is solid, then $(\mathcal{F} \ltimes \mathcal{E}_1, tv)$ is also solid.*

Proof. By 8.16 (1) and 8.17, the inclusion functor $(\mathcal{F} \ltimes \mathcal{E}_1)^{tv} \hookrightarrow (\mathcal{F} \ltimes \mathcal{E}_2)^{tv}$ satisfies the resolution condition and $(\mathcal{F} \ltimes \mathcal{E}_2, tv)$ is solid. Hence $(\mathcal{F} \ltimes \mathcal{E}_1, tv)$ is also solid by 5.4. \square

Next theorem is an abstraction of Corollary 5.14 in [Moc13].

Theorem 8.19. *Let $2 \leq n$ be a positive integer and we put $S = (n)$. Assume that w is compatible with \mathfrak{F} .*

(1) *Assume that for any integer $0 \leq k \leq n-1$, there exists a strict exact subcategory x_k of $\mathbf{Cub}^{(n-k-1)} \mathcal{A}$ such that a triple $(\ltimes \mathfrak{F}|_{(n-k-1)}^{[n-k+1, n]}, x_k, \ltimes \mathfrak{F}|_{(n-k-1)}^{[n-k, n]})$ is an adroit (resp. a strongly adroit) system of $\mathbf{Cub}^{(n-k-1)} \mathcal{A}$ (resp. and (\mathcal{F}_S, w) is a Waldhausen exact category). Then the relative exact functor $H_{(n-k)}^{[n-k+1, n], \{n-k\}} : (\ltimes \mathfrak{F}|_{(n-k)}^{[n-k+1, n]}, tw) \rightarrow (\ltimes \mathfrak{F}|_{(n-k-1)}^{[n-k, n]}, tw)$ is a derived equivalence (resp. K^W -equivalence). In particular the relative exact functor $H_0^S : (\ltimes \mathfrak{F}, tw) \rightarrow (\mathcal{F}_S, w)$ is a derived equivalence (resp. K^W -equivalence).*

(2) *Assume that for any integer $1 \leq k \leq n-1$, there exists strict exact subcategory y_k of $\mathbf{Cub}^{(k)} \mathcal{A}$ such that $(\ltimes \mathfrak{F}|_{(k)}^{\emptyset}, y_k, \ltimes \mathfrak{F}|_{(k)}^{\{k+1\}})$ is an adroit system. Then*

(i) *There is a derived right flag of $\ltimes \mathfrak{F}$,*

$$0 \rightarrow \ltimes \mathfrak{F}|_{(1)}^{\emptyset} \xrightarrow{\text{ext}_{(1), \{2\}}^{\emptyset}} \ltimes \mathfrak{F}|_{(2)}^{\emptyset} \xrightarrow{\text{ext}_{(2), \{3\}}^{\emptyset}} \dots \xrightarrow{\text{ext}_{(n-1), \{n\}}^{\emptyset}} \ltimes \mathfrak{F}|_{(n)}^{\emptyset} = \ltimes \mathfrak{F}.$$

(ii) *Moreover if (\mathcal{F}_S, w) is solid, then $(\ltimes \mathfrak{F}, tw)$ is also solid.*

(3) *Let \mathcal{R} be a reasonable subcategory of $\mathbf{RelEx}_{\text{strict}}$ such that for any disjoint pair of subsets U and V of (n) , $(\ltimes \mathfrak{F}|_U^V, tw)$ is in \mathcal{R} and F a localizing theory on \mathcal{R} (resp. $F = K^W$). Assume that for any integer $1 \leq k \leq n-1$, there exists strict exact subcategory z_k of $\mathbf{Cub}^{(k)} \mathcal{A}$ such that $(\ltimes \mathfrak{F}|_{(k)}^{\emptyset}, z_k, \ltimes \mathfrak{F}|_{(k)}^{\{k+1\}})$ is an adroit system (resp. and (\mathcal{F}_S, w) is a Waldhausen exact*

category). Then

(i) The exact functors

$$\lambda_n : \ltimes \mathfrak{F} \rightarrow \prod_{T \in \mathcal{P}(S)} \mathcal{F}_T, \quad x \mapsto (H_0^T(x)_\emptyset)_T \quad \text{and}$$

$$\lambda'_n : \ltimes \mathfrak{F}^{tw} \rightarrow \mathcal{F}_S^w \times \prod_{T \in \mathcal{P}(S) \setminus \{S\}} \mathcal{F}_T, \quad x \mapsto (H_0^T(x)_\emptyset)_T$$

are F -equivalences.

(ii) Moreover if (\mathcal{F}_S, w) satisfies the F -fibration axiom, then $(\ltimes \mathfrak{F}, tw)$ also satisfies the F -fibration axiom.

Proof. (1) It is just a consequence of 8.16 (resp. [Moc13, 2.21]) (2).

(2) We proceed by induction on n . (i) follows from 8.16 (3) and (ii) is a consequence of 8.18.

(3) (i) We proceed by induction on n . For any $1 \leq k \leq n-1$, we define

$$\underline{\lambda}_k : \ltimes \mathfrak{F}|_{[k]}^{\{k+1\}} \rightarrow \prod_{T \in \mathcal{P}([k])} \mathcal{F}_{T \sqcup \{k+1\}}$$

to be an exact functor by sending an object x in $\ltimes \mathfrak{F}|_{[k]}^{\{k+1\}}$ to $(H_0^{T \sqcup \{k+1\}}(x))_T$. We consider the following commutative diagram.

$$\begin{array}{ccccc} F(\ltimes \mathfrak{F}|_{[k]}^\emptyset) & \xrightarrow{\text{II}} & F(\ltimes \mathfrak{F}|_{[k-1]}^{\{k-1\}}) \oplus F(\ltimes \mathfrak{F}|_{[k-1]}^\emptyset) & \xrightarrow{\text{III}} & \bigoplus_{T \in \mathcal{P}(S)} F(\mathcal{F}_T) \\ \downarrow & & \downarrow & & \\ F((\ltimes \mathfrak{F}|_{[k-1]}^{\{k\}}) \ltimes z_k) & \xrightarrow{\text{I}} & F(\ltimes \mathfrak{F}|_{[k-1]}^{\{k\}}) \oplus F(z_k) & & \end{array}$$

where the vertical lines are induced from the inclusion functors and the maps **I**, **II** and **III** are $\begin{pmatrix} F(H_{[k]}^{\emptyset, \{k\}}) \\ F(\text{res}_{[k]}^{\emptyset, 1}) \end{pmatrix}$, $\begin{pmatrix} F(H_{[k]}^{\emptyset, \{k\}}) \\ F(\text{res}_{[k]}^{\emptyset, 1}) \end{pmatrix}$ and $(F(\underline{\lambda}_{k-1}) \quad F(\lambda_{k-1}))$ respectively. Notice that the composition the map **II** with the map **III** is just the map $F(\lambda_n)$. Then since the sequence

$$(z_k, i_{z_k}) \rightarrow ((\ltimes \mathfrak{F}|_{[k-1]}^{\{k\}}) \ltimes z_k, i_{(\ltimes \mathfrak{F}|_{[k-1]}^{\{k\}}) \ltimes z_k}) \xrightarrow{H_0} (\ltimes \mathfrak{F}|_{[k-1]}^{\{k\}}, i_{\ltimes \mathfrak{F}|_{[k-1]}^{\{k\}}})$$

is a right quasi-split exact sequence by [Moc13, 2.19], the map **I** is a homotopy equivalence of spectra by 7.10 (3). Since the vertical lines are homotopy equivalences of spectra by 8.16 (1), it turns out that the map **II** is also. Hence we obtain the result for $n = 2$. By inductive hypothesis, the map **III** is also a homotopy equivalence of spectra and we obtain the result.

For the second assertion, we just apply $\mathfrak{F}' = \{\mathcal{F}_T\}_{T \in \mathcal{P}(S) \setminus \{S\}} \sqcup \{\mathcal{F}_S^w\}$ to the first sentence. Notice that by virtue of 8.13 and 8.15, \mathfrak{F}' satisfies the assumption of (3) (i).

(ii) By the commutative diagram below

$$\begin{array}{ccccc} F((\ltimes \mathfrak{F})^{tw}) & \xrightarrow{\quad} & F(\ltimes \mathfrak{F}) & \xrightarrow{\quad} & F(\ltimes \mathfrak{F}, tw) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \bigoplus_{T \in \mathcal{P}(S) \setminus \{S\}} F(\mathcal{F}_T) \oplus F(\mathcal{F}_S^w) & \xrightarrow{\quad} & \bigoplus_{T \in \mathcal{P}(S)} F(\mathcal{F}_T) & \xrightarrow{\quad} & F(\mathcal{F}_S, tw), \end{array}$$

it turns out that $(\ltimes \mathfrak{F}, tw)$ also satisfies the F -fibration axiom. \square

Remark 8.20. For an abelian category \mathcal{A} , we put $\mathbf{A} = (\mathcal{A} \ltimes \mathcal{A}, \text{qis})$. Then \mathbf{A} satisfies K^W -fibration axiom by 8.19 (3). But \mathbf{A} does not satisfy the factorization axiom. For assume that for an non acyclic complex x in $\mathcal{A} \ltimes \mathcal{A}$, if the morphism $x \rightarrow 0$ admits a factorization $x \xrightarrow{i} y \xrightarrow{u} 0$ with $u \in \text{qis}$ and then we need to have $0 \neq H_0(x) \xrightarrow{H_0(i)} H_0(y) = 0$. It is a contradiction.

9 Koszul cubes

In this section, we prove that the category of Koszul cubes with the class of total quasi-isomorphisms is both \mathbb{K} - and K^W -excellent in 9.6 and it is derived equivalent to the complicial Waldhausen category of perfect complexes in 9.7. In this section, fix a non-empty finite set S and we denote the category of finitely generated A -modules by \mathcal{M}_A . We start by reviewing the notion A -sequences.

Let $\{f_s\}_{s \in S}$ be a family of elements in A . We say that the sequence $\{f_s\}_{s \in S}$ is an A -**sequence** if $\{f_s\}_{s \in S}$ forms an A -regular sequences in any order. Fix an A -sequence $\mathfrak{f}_S = \{f_s\}_{s \in S}$. For any subset T , we denote the family $\{f_t\}_{t \in T}$ by \mathfrak{f}_T .

Definition 9.1 (Koszul cube). (cf. [Moc13, 4.8]) A **Koszul cube** x associated with an A -sequence $\mathfrak{f}_S = \{f_s\}_{s \in S}$ is an S -cube in \mathcal{P}_A the category of finitely generated projective A -modules such that for each subset T of S and k in T , d_T^k is an injection and $f_k^{m_k} \text{Coker } d_T^k = 0$ for some m_k . We denote the full subcategory of $\mathbf{Cub}^S \mathcal{P}_A$ consisting of those Koszul cubes associated with \mathfrak{f}_S by $\mathbf{Kos}_A^{\mathfrak{f}_S}$.

Recall the notation of $\mathcal{M}_A^I(q)$ from Conventions (4) (iii). The category of Koszul cubes is described in terms of multi-semi direct products of Quillen exact categories $\mathcal{M}_A^{\mathfrak{f}_T}(\#T)$ as in 9.2.

Theorem 9.2. (cf. [Moc13, 4.20]). *We have the equality*

$$\mathbf{Kos}_A^{\mathfrak{f}_S} = \ltimes_{T \in \mathcal{P}(S)} \mathcal{M}_A^{\mathfrak{f}_T}(\#T).$$

The description of the category of Koszul cubes above gives a motivation to define the following categories. In the rest of this section, fix a disjoint decomposition $S = U \sqcup V$.

9.3. For any non-negative integer p , we define the category $\mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_V)(p)$ which is a full subcategory of $\mathbf{Cub}^V \mathcal{M}_A$ by

$$\mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_V)(p) = \ltimes_{T \in \mathcal{P}(V)} \mathcal{M}_A^{\mathfrak{f}_{T \cup U}}(p + \#T).$$

Then $\mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_V)(p)$ is closed under extensions in $\mathbf{Cub}^V \mathcal{M}_A$. In particular it becomes a Quillen exact category in the natural way. We write $\text{tq}(\mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_V)(p))$ or shortly tq for the class of total weak equivalences in $\mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_V)(p)$ associated with the class of all isomorphisms in $\mathcal{M}_A^{\mathfrak{f}_S}(p + \#S)$. A morphism in tq is said to be a **total quasi-isomorphism**. Note that we have the equalities

$$\begin{aligned} \mathcal{M}_A(\mathfrak{f}_\emptyset; \mathfrak{f}_S)(0) &= \mathbf{Kos}_A^{\mathfrak{f}_S} \quad \text{and} \\ \mathcal{M}_A(\mathfrak{f}_S; \mathfrak{f}_\emptyset)(p) &= \mathcal{M}_A^{\mathfrak{f}_S}(p). \end{aligned}$$

In the rest of this section, let $p \geq \#U$ be an integer and we put $\mathfrak{F} = \{\mathcal{M}_A^{\mathfrak{f}_U \sqcup T}(p + \#T)\}_{T \in \mathcal{P}(V)}$. Recall the definition of the resolution conditions from 5.2. The following proposition is essentially proven in [Moc13, 5.9, 5.11]

Proposition 9.4. *The inclusion functors*

$$i : \mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_V)(p) \hookrightarrow \mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_V)(p+1) \quad \text{and} \\ i' : \mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_V)(p)^{\text{tq}} \hookrightarrow \mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_V)(p+1)^{\text{tq}}$$

satisfy the resolution conditions. □

Recall the definition of the (strongly) adroit systems from 8.14.

Theorem 9.5. (cf. [Moc13, 5.13]). *Assume that V is a non-empty set. Then the triple*

$$(\mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_{V \setminus \{v\}})(p), \mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_{V \setminus \{v\}})(p+1), \mathcal{M}_A(\mathfrak{f}_{U \sqcup \{v\}}; \mathfrak{f}_{V \setminus \{v\}})(p+1))$$

is a strongly adroit system in $\mathbf{Cub}^{V \setminus \{v\}} \mathcal{M}_A$ for any $v \in V$.

In the rest of this section, for simplicity we put $\mathcal{H} = \mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_V)(p)$. Recall the definitions of the flag from 6.11 and of the excellent relative exact categories from 7.4.

Corollary 9.6. (1) *(\mathcal{H}, tq) is a very strict solid Waldhausen exact category and satisfies the K^W -fibration axiom. In particular it is both K^W - and \mathbb{K} -excellent.*

(2) *The exact functor $H_0 \text{Tot} : (\mathcal{H}, \text{tq}) \rightarrow (\mathcal{M}_A^{\mathfrak{f}_S}(p + \#V), i)$ is a derived equivalence. In particular, the exact functor $H_0^S : (\mathbf{Kos}_A^{\mathfrak{f}_S}, \text{tq}) \rightarrow (\mathcal{M}_A^{\mathfrak{f}_S}(\#S), i)$ is a derived equivalence.*

(3) *Let $V = \{v_1, v_2, \dots, v_r\}$ and we put $V_k = \{v_i; 1 \leq i \leq k\}$ for any $1 \leq k \leq r$ and $V_0 = \emptyset$ and $e_k := \text{ext}^{\emptyset, \{k+1\}} : \mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_{V_k})(p) \rightarrow \mathfrak{F}|_{\emptyset}^{\{k\}} \rightarrow \mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_{V_{k+1}})(p)$. Then the sequence*

$$0 \rightarrow \mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_{V_0})(p) \xrightarrow{e_0} \mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_{V_1})(p) \xrightarrow{e_1} \dots \xrightarrow{e_{r-1}} \mathcal{H}$$

is a derived right flag.

(4) *The inclusion functor $\mathcal{H}^{\text{tq}} \hookrightarrow \mathcal{H}$ and the identity functor of \mathcal{H} induce a derived right quasi-split exact sequence*

$$(\mathcal{H}^{\text{tq}}, i_{\mathcal{H}^{\text{tq}}}) \rightarrow (\mathcal{H}, i_{\mathcal{H}}) \rightarrow (\mathcal{H}, \text{tq}).$$

Proof. First notice that the class of all isomorphisms in $\mathcal{M}_A^{\mathfrak{f}_S}(p + \#S)$ is compatible with \mathfrak{F} and (\mathcal{H}, tq) is a Waldhausen exact category which satisfies the extensional axiom by 8.11 and 8.13. Assertions (2) and (3) and the fact that (\mathcal{H}, tq) satisfies the both the solid and the K^W -fibration axioms follow from 8.19 and 9.5. The fact that (\mathcal{H}, tq) is very strict is just a consequence of (4). The last assertion of (1) follows from 7.5.

Proof of assertion (4): By virtue of (2), we only need to check that the sequence

$$(\mathcal{H}^{\text{tq}}, i_{\mathcal{H}^{\text{tq}}}) \rightarrow (\mathcal{H}, i_{\mathcal{H}}) \xrightarrow{H_0^V} (\mathcal{M}_A^{\mathfrak{f}_S}(p + \#V), i_{\mathcal{M}_A^{\mathfrak{f}_S}(p + \#V)})$$

is a derived right quasi-split exact sequence. For simplicity we put $\mathcal{H}' := \mathcal{M}_A(\mathfrak{f}_U; \mathfrak{f}_V)(p + \#V)$ and let $I : \mathcal{H} \hookrightarrow \mathcal{H}'$ and $J : \mathcal{H}^{\text{tq}} \hookrightarrow \mathcal{H}'^{\text{tq}}$ be the inclusion functors. We define $s : \mathcal{M}_A^{\mathfrak{f}_S}(p + \#V) \rightarrow \mathcal{H}'$ to be an exact functor by sending an object x to $s(x)$ where $s(x)_T$ is x if $T = \emptyset$ and 0 if $T \neq \emptyset$. There is a natural transformation $C : I \rightarrow s H_0^V$ such that for any object x in \mathcal{H} ,

$$(Cx)_{\emptyset} : x_{\emptyset} \rightarrow s H_0^V(x) = \text{Coker} \left(\bigoplus_{v \in V} x_{\{v\}} \rightarrow x_{\emptyset} \right)$$

is the natural quotient morphism. We define the exact functor $r : \mathcal{H} \rightarrow \mathcal{H}'^{\text{tq}}$ by $r := \ker(I \xrightarrow{C} s H_0^V)$. We illustrate the situation with the commutative diagram below.

$$\begin{array}{ccccc}
 \mathcal{H}^{\text{tq}} & \xrightarrow{i} & \mathcal{H} & \xrightarrow{H_0^V} & \mathcal{M}_A^{\text{fs}}(p + \#V) \\
 \downarrow J & \swarrow r & \downarrow I & \swarrow s & \\
 \mathcal{H}'^{\text{tq}} & \xrightarrow{i'} & \mathcal{H}' & & .
 \end{array}$$

By definition, there exists an admissible exact sequence of exact functors from \mathcal{H} to \mathcal{H}'^{tq}

$$i' r \xrightarrow{A} I \xrightarrow{C} s H_0.$$

Since I and J are derived equivalences by 9.4, the pair (r, s) yields a right quasi-splitting of a sequence

$$\mathcal{D}_b(\mathcal{H}^{\text{tq}}) \xrightarrow{\mathcal{D}_b(i)} \mathcal{D}_b(\mathcal{H}) \xrightarrow{\mathcal{D}_b(H_0)} \mathcal{D}_b(\mathcal{M}_A^{\text{fs}}(p + \#V)).$$

Hence we obtain the result. \square

Recall the definition of $\mathbf{Perf}_{\text{Spec } A}^{V(\text{fs})}$ from the introduction.

9.7 (Proof of Theorem 0.5). Let $j' : \mathcal{M}_A^{\text{fs}}(\#S) \rightarrow \mathbf{Perf}_{\text{Spec } A}^{V(\text{fs})}$ denote the composition of $j_{\mathcal{M}_A^{\text{fs}}(\#S)} : \mathcal{M}_A^{\text{fs}}(\#S) \rightarrow \mathbf{Ch}_b(\mathcal{M}_A^{\text{fs}}(\#S))$ and the inclusion functor $\mathbf{Ch}_b(\mathcal{M}_A^{\text{fs}}(\#S)) \hookrightarrow \mathbf{Perf}_{\text{Spec } A}^{V(\text{fs})}$. There exists a canonical relative natural equivalence $\text{Tot} \rightarrow j' H_0 \text{Tot}$ induced by the quotient map $(\text{Tot } x)_0 \twoheadrightarrow H_0 \text{Tot } x$ for any x in $\mathbf{Kos}_A^{\text{fs}}$. Hence $\mathcal{D}_b \text{Tot} = \mathcal{D}_b j' \mathcal{D}_b H_0 \text{Tot}$ by 3.21 (2) and the functor $\mathcal{D}_b H_0 \text{Tot}$ and $\mathcal{D}_b j'$ are equivalences of triangulated categories by 9.6 (2), and 4.15 (2) and [HM10, 3.3] respectively. Therefore we obtain the result. \square

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